CHAPTER 3

Euler and Infinite Series

When the seventeenth century dawned, infinite series were little understood and infrequently encountered. By the century's end, a vast body of specific examples and general theorems had been developed. Jakob Bernoulli's *Tractatus de seriebus infinitis* of 1689, mentioned in the previous chapter, presented a state-of-the-art account of this explosion of knowledge. It was an exciting time, and mathematicians had reason to be proud of their progress over the past hundred years.

Such achievements notwithstanding, there were major problems that defied solution and thus served as conspicuous challenges to scholars of the coming century. Euler, of course, was such a scholar, and in one famous case—the so-called "Basel problem"—he rose to the challenge in spectacular fashion. In this chapter we tell the story of his mathematical triumph.

Prologue

Jakob Bernoulli loved infinite series. Not only did he prove the divergence of the harmonic series, but he also knew exact sums for a number of convergent ones. Simplest among these was the the summation formula for the infinite geometric series:

$$a + ar + ar^2 + \cdots + ar^k + \cdots = \frac{a}{1-r}$$

provided $-1 < r < 1$.

Other, more sophisticated examples could be summed as well. For instance, consider $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \cdots$, where the kth denominator is the so-called kth triangular number, $k(k+1)/2$. A seventeenth-century evaluation of this series was short and sweet:

$$\begin{aligned}1 &+ \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \cdots \\ &= 2\left[\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots\right] \\ &= 2\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots\right] \\ &= 2[1] = 2,\end{aligned}$$

because, within the second set of square brackets, all terms but the first cancel one another. Students of calculus should recognize

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)/2} = 2$$

as a well-known "telescoping series."

Less familiar is Jakob Bernoulli's summation of the infinite series

$$\frac{a}{b} + \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \frac{a+3c}{bd^3} + \cdots,$$

whose numerators form an arithmetic progression

$$a, a+c, a+2c, a+3c, \ldots$$

and whose corresponding denominators form a geometric progression

$$b, bd, bd^2, bd^3, \ldots.$$

For instance, if $a = 1$, $b = 3$, $c = 5$, and $d = 7$, we have

$$\frac{1}{3} + \frac{6}{21} + \frac{11}{147} + \frac{16}{1029} + \frac{21}{7203} + \frac{26}{50421} + \cdots,$$

whose exact sum is far from obvious.

In Section XIV of the *Tractatus*, Jakob evaluated this series.[1] His insight was to decompose it as follows:

$$\begin{aligned}\frac{a}{b} &+ \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \frac{a+3c}{bd^3} + \cdots \\ &= \left(\frac{a}{b} + \frac{a}{bd} + \frac{a}{bd^2} + \frac{a}{bd^3} + \cdots\right)\end{aligned}$$

[1] Jakob Bernoulli, p 247

$$
\begin{array}{llllllll}
+ \left(\frac{c}{bd} + \right. & \frac{c}{bd^2} + & \frac{c}{bd^3} + & \left.\cdots\right) \\
& + \left(\frac{c}{bd^2} + \right. & \frac{c}{bd^3} + & \left.\cdots\right) \\
& & + \left(\frac{c}{bd^3} + \right. & \left.\cdots\right) \\
\vdots \quad \vdots \quad \vdots & \vdots & \vdots \quad \vdots & \vdots
\end{array}
$$

Each infinite series in parentheses is geometric and, provided $d > 1$, convergent. Replacing these by their sums gives:

$$
\begin{aligned}
\frac{a}{b} &+ \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \frac{a+3c}{bd^3} + \cdots \\
&= \frac{a/b}{1-1/d} + \frac{c/bd}{1-1/d} + \frac{c/bd^2}{1-1/d} + \cdots \\
&= \frac{ad}{bd-b} + \frac{c}{b(d-1)}\left[1 + \frac{1}{d} + \frac{1}{d^2} + \frac{1}{d^3} + \cdots\right] \\
&= \frac{ad}{bd-b} + \frac{c}{b(d-1)}\left[\frac{1}{1-1/d}\right] = \frac{ad^2 - ad + cd}{bd^2 - 2bd + b},
\end{aligned}
$$

because the series in square brackets is geometric as well.

So, for the example above, we have:

$$
\frac{1}{3} + \frac{6}{21} + \frac{11}{147} + \frac{16}{1029} + \frac{21}{7203} + \frac{26}{50421} + \cdots = \frac{77}{108}.
$$

And there were others. For instance, Jakob found that

$$
\sum_{k=1}^{\infty} \frac{k^2}{2^k} = 6
$$

and

$$
\sum_{k=1}^{\infty} \frac{k^3}{2^k} = 26
$$

(which remain good problems to this day).[2] With each success, he must have felt ever more confident of his powers.

Eventually he turned his attention to series of the form

$$
\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{k^p} + \cdots,
$$

[2] Ibid., pp. 248–249.

which are today called "p-series" for obvious reasons. If $p = 1$, we have the (divergent) harmonic series which Jakob had handled perfectly. But what if $p = 2$? What is the exact sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{k^2} + \cdots ?$$

The problem was not a new one. Decades earlier, Pietro Mengoli had raised the question and found himself unable to determine this sum. The same could be said for Leibniz, inventor of calculus and master of so many infinite series. Now it was Jakob Bernoulli's turn. One imagines his growing frustration with a series that, on the face of it, seemed no more difficult than those whose secrets he had previously uncovered.

This is not to say that progress was nonexistent. By employing the inequality $2k^2 \geq k(k+1)$, Bernoulli recognized that

$$\frac{1}{k^2} \leq \frac{1}{k(k+1)/2},$$

and thus

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{k^2} + \cdots \leq 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{1}{k(k+1)/2} + \cdots,$$

where this latter (telescoping) series converges to 2, as seen above. Because the larger series has a finite sum, Bernoulli reasoned that the smaller one must as well. More explicitly, it was clear that $\sum_{k=1}^{\infty} 1/k^2 \leq 2$. And because $1/k^p \leq 1/k^2$ for all $p \geq 2$, the same argument established that $\sum_{k=1}^{\infty} 1/k^p$ converges for $p = 3, 4, 5, \ldots$.

This stands as an early—and nicely done—example of what is now called the "comparison test" for series convergence. For all of its cleverness, however, it did not provide an *exact* sum for the series in question. On this more difficult matter, Jakob admitted defeat. Writing from Basel, he included in the *Tractatus* his plea for help:

> If anyone finds and communicates to us that which thus far has eluded our efforts, great will be our gratitude.[3]

With these words, the mathematical community was handed a formal, and formidable, challenge. In the end, the "Basel problem" would outlive Jakob Bernoulli and the century that spawned it. Only in the eighteenth century did this great problem meet its match.

[3] Ibid., p. 254.

Enter Euler

It is not clear exactly when Euler first considered the matter, but by 1731, at the age of 24, he was hard at work upon it. It occurred to him, as it had to others before, that a reasonable first step would be to approximate the infinite series $\sum_{k=1}^{\infty} 1/k^2$ by adding the first few—or few hundred—terms. Unfortunately, because this series converges so slowly, such frontal attacks are not particularly illuminating. For instance,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.54977 \text{ (ten terms);}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{10000} \approx 1.63498 \text{ (one hundred terms);}$$

and

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{1000^2} \approx 1.64393 \text{(one thousand terms).}$$

We now know that, in spite of its prodigious number of terms, this last result is correct to only *two* decimal places. Other than the comforting fact that all these partial sums remain below 2.000 (as Bernoulli's comparison test had proved), direct numerical approximation is of little value.

Then, in a 1731 paper, the young Euler found a way to improve dramatically such numerical approximations. His discovery, remarkable in its vision and fearless in its manipulation of symbols, was truly ingenious.[4]

Euler's trick was to evaluate the (improper) integral

$$I = \int_0^{1/2} -\frac{\ln(1-t)}{t}\, dt$$

in two different ways. On the one hand, he replaced $\ln(1-t)$ by its series expansion (see Chapter 2) and integrated termwise to get:

$$\begin{aligned} I &= \int_0^{1/2} -\frac{-t - t^2/2 - t^3/3 - t^4/4 - \cdots}{t}\, dt \\ &= \int_0^{1/2} \left(1 + \frac{t}{2} + \frac{t^2}{3} + \frac{t^3}{4} + \cdots\right) dt \\ &= t + \frac{t^2}{4} + \frac{t^3}{9} + \frac{t^4}{16} + \cdots \Big|_0^{1/2} \end{aligned}$$

[4]Euler, *Opera Omnia*, Ser. 1, Vol 14, pp 39–41.

$$= \frac{1}{2} + \frac{1/2^2}{4} + \frac{1/2^3}{9} + \frac{1/2^4}{16} + \cdots. \quad (3.1)$$

On the other hand, he substituted $z = 1 - t$ to transform the original integral as follows:

$$I = \int_0^{1/2} -\frac{\ln(1-t)}{t}\,dt = \int_1^{1/2} \frac{\ln z}{1-z}\,dz$$

$$= \int_1^{1/2} (1 + z + z^2 + z^3 + \cdots)\ln z\,dz$$

$$= \int_1^{1/2} \ln z\,dz + \int_1^{1/2} z\ln z\,dz + \int_1^{1/2} z^2\ln z\,dz + \int_1^{1/2} z^3\ln z\,dz + \cdots,$$

because $1/(1-z)$ is the sum of the geometric series $1 + z + z^2 + z^3 + \cdots$.

Integration by parts implies that

$$\int_1^{1/2} z^n \ln z\,dz = \frac{z^{n+1}}{n+1}\ln z - \frac{z^{n+1}}{(n+1)^2}\bigg|_1^{1/2},$$

and so this last expression becomes:

$$I = (z\ln z - z) + \left(\frac{z^2}{2}\ln z - \frac{z^2}{4}\right) + \left(\frac{z^3}{3}\ln z - \frac{z^3}{9}\right)$$

$$+ \left(\frac{z^4}{4}\ln z - \frac{z^4}{16}\right) + \cdots\bigg|_1^{1/2}$$

$$= \ln z\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots\right] - \left(z + \frac{z^2}{4} + \frac{z^3}{9} + \frac{z^4}{16} + \cdots\right)\bigg|_1^{1/2}$$

$$= \ln z\big[-\ln(1-z)\big] - \left(z + \frac{z^2}{4} + \frac{z^3}{9} + \frac{z^4}{16} + \cdots\right)\bigg|_1^{1/2}$$

$$= -\left[\ln\left(\frac{1}{2}\right)\right]^2 - \left(\frac{1}{2} + \frac{1/2^2}{4} + \frac{1/2^3}{9} + \frac{1/2^4}{16} + \cdots\right)$$

$$+ [\ln 1][\ln 0] + \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Euler simply discarded $[\ln 1][\ln 0]$, although the modern reader might prefer to invoke l'Hôpital's Rule to verify that $\lim_{z\to 1^-}[\ln z][\ln(1-z)] = 0$. In any case, he arrived at:

$$I = -[\ln 2]^2 - \left(\frac{1}{2} + \frac{1/2^2}{4} + \frac{1/2^3}{9} + \frac{1/2^4}{16} + \cdots\right) + \sum_{k=1}^{\infty} \frac{1}{k^2}. \qquad (3.2)$$

Then he equated the expressions for I in (3.1) and (3.2) and solved:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 2\left(\frac{1}{2} + \frac{1/2^2}{4} + \frac{1/2^3}{9} + \frac{1/2^4}{16} + \cdots\right) + [\ln 2]^2$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2 2^{k-1}} + [\ln 2]^2.$$

By this time the reader must have noticed a number of symbolic manipulations that require careful handling. Euler paid no heed to such matters as the existence of improper integrals or the termwise integration of infinite series. Nevertheless, his fusion of the log series, the geometric series, and integration by parts—all with the object of reaching an alternative expression for $\sum_{k=1}^{\infty} 1/k^2$—was a masterstroke. What made this effort worthwhile was that the resulting formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2 2^{k-1}} + [\ln 2]^2$$

consists of a rapidly converging series (thanks to the 2^{k-1} term in the denominator) along with the number $[\ln 2]^2$, which Euler knew to dozens of decimal places. Using only fourteen terms of this new formula, one finds that $\sum_{k=1}^{\infty} 1/k^2 \approx 1.644934$, an answer correct to six places. This is far more accurate than summing a *thousand* terms of the original series. Euler's ingenuity had paid off.

Or had it? In spite of this vastly improved estimate, it was still just an estimate. Jakob Bernoulli, one remembers, had challenged the world to find the exact sum. In this sense, the problem seemed as far from resolution as ever.

But the end was in sight. Four years later, in 1735, Euler finally succeeded where so many others had failed. Admitting that his previous efforts had fallen short and that "it seemed most unlikely to be able to find anything new about this," Euler wrote with obvious joy:[5]

> Now, however, against all expectation I have found an elegant expression for the sum of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ + etc., which depends on the quadrature of the circle... I have found that six times the sum

[5]Ibid., pp 73–74

of this series is equal to the square of the circumference of a circle whose diameter is 1.

To us, this wording about diameters and circumferences seems roundabout—both geometrically and metaphorically—but because the circumference of such a circle has length π, Euler was asserting (in modern notation) that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Ever since, this has stood as one of the most wonderful formulas in mathematics. Those seeing it for the first time are puzzled by the unexpected appearance of π in a series of perfect squares, and at first glance it looks more like a typo than a theorem. Never fear: Euler was right.

His brief argument required two modest observations and one typically Eulerian leap of faith. First of all, we note that if $P(x) = 0$ is an nth degree polynomial equation with non-zero roots $a_1, a_2, a_3, \ldots, a_n$ and such that $P(0) = 1$, then in factored form

$$P(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\left(1 - \frac{x}{a_3}\right)\cdots\left(1 - \frac{x}{a_n}\right).$$

This is self-evident, because substituting $x = 0$ gives $P(0) = 1$, just as substituting $x = a_k$ yields $P(a_k) = 0$ for $k = 1, 2, \ldots n$.

Second, he needed the series expansion of $\sin x$, namely

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots.$$

This formula, recognizable to any calculus student, was well known to Euler. (In Chapter 5, we shall discuss his derivation of this expansion, one whose use of the infinitely large and infinitely small is reminiscent of his development of the series for $\log(1 + x)$ from Chapter 2.)

These were the prerequisites underpinning his great discovery. The leap of faith was a belief that whatever holds for an ordinary polynomial will likewise hold for an "infinite polynomial." In this case, he assumed that a polynomial-like expression with infinitely many roots can be factored as $P(x)$ was factored above. Euler offered no proof of this, but for one who believed in the universality of formulas, it was a natural symbolic extension.

We now are ready for Euler's solution of the Basel problem.[6]

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[6]Ibid., pp. 84–85

Theorem. $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Proof. Euler introduced

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots,$$

which he regarded as an "infinite polynomial." Clearly $P(0) = 1$. To find the roots of $P(x) = 0$, note that for $x \neq 0$,

$$\begin{aligned} P(x) &= x\left[\frac{1 - x^2/3! + x^4/5! - x^6/7! + x^8/9! - \cdots}{x}\right] \\ &= \frac{x - x^3/3! + x^5/5! - x^7/7! + x^9/9! - \cdots}{x} = \frac{\sin x}{x}. \end{aligned}$$

So $P(x) = 0$ implies that $\sin x = 0$, which means in turn that $x = \pm k\pi$ for $k = 1, 2, \ldots$. Note that $x = 0$ is *not* a solution to $P(x) = 0$ because $P(0) = 1$.

In light of the observation above, he now factored $P(x)$ as:

$$\begin{aligned} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots &= P(x) \\ &= \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right) \\ &\quad \times \left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right)\cdots \\ &= \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\left[1 - \frac{x^2}{16\pi^2}\right]\cdots. \end{aligned} \tag{3.3}$$

This is the chapter's most important formula. Euler had written $P(x)$ in two very different ways, equating the infinite sum on the left with the infinite product on the right.

What next? For Euler, nothing could be more natural than to expand the right side of (3.3) to get:

$$\begin{aligned} 1 &- \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots \\ &= 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right)x^2 + \cdots \end{aligned} \tag{3.4}$$

where the coefficients of x^4 and higher (even) powers are unnecessary and, for the moment, unknown. He then equated the coefficients of x^2 in (3.4) to get

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right)$$
$$= -\frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right),$$

and concluded in dramatic fashion that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}. \qquad \text{Q.E.D.}$$

As he had promised, six times the sum of the series is the square of π. The Basel Problem was solved.

Of course Euler stands open to the charge of playing fast and loose with the logic. Over time, even *he* appeared troubled by the course his argument had taken and in later writings provided alternative—and what he considered more rigorous—derivations of this same formula. We shall examine one of these in the chapter's epilogue. Although none was entirely successful by modern standards, the reader should be assured that fully rigorous proofs have subsequently confirmed Euler's result.[7]

Such misgivings aside, Euler was confident that he had answered Bernoulli's unresolved question. There were internal indications that bolstered this certainty. For instance, a quick calculation revealed that $\pi^2/6 \approx 1.644934$, the precise estimate Euler had discovered a few years earlier. Numerically, he was right on target.

Moreover, his line of reasoning led to a previously known gem: Wallis's formula. In 1655, the English mathematician John Wallis (1616–1703), considering a different question and following a different logical path, had demonstrated that

$$\frac{2}{\pi} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}.$$

In the *Introductio*, Euler showed how the infinite product of (3.3) led to an alternate derivation of Wallis's formula. Putting $x = \pi/2$ into that expression yields

$$P\left(\frac{\pi}{2}\right) = \left[1 - \frac{(\pi/2)^2}{\pi^2}\right]\left[1 - \frac{(\pi/2)^2}{4\pi^2}\right]\left[1 - \frac{(\pi/2)^2}{9\pi^2}\right]\left[1 - \frac{(\pi/2)^2}{16\pi^2}\right]\cdots,$$

[7]Dan Kalman, "Six Ways to Sum a Series," *The College Mathematics Journal*, Vol 24, No 5, 1993, pp. 402–421

which simplifies to

$$\frac{\sin(\pi/2)}{\pi/2} = \left[1 - \frac{1}{4}\right]\left[1 - \frac{1}{16}\right]\left[1 - \frac{1}{36}\right]\left[1 - \frac{1}{64}\right]\cdots$$
$$= \frac{3}{4} \times \frac{15}{16} \times \frac{35}{36} \times \cdots.$$

In short,

$$\frac{2}{\pi} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}.$$

Here we have Wallis's formula as a corollary. Surely this established that Euler's train of thought had not derailed. If his argument could recover previously known results such as this, there seemed all the more reason to embrace his initial conclusion.[8]

Quickly Euler's discovery flashed around the European mathematical community (if "flashed" is the correct verb to characterize eighteenth-century mail service). When Johann Bernoulli learned of the solution he wrote:

> *Utinam Frater superstes effet* !
> (If only my brother were alive!)[9]

André Weil called this "One of Euler's most sensational early discoveries, perhaps the one which established his growing reputation most firmly."[10] After this triumph, anyone who counted in European mathematics knew of the young genius who had succeeded so brilliantly where all others had failed.

It is easy to imagine that such success would lead many people to sit back, accept the plaudits of colleagues, and live off their well-deserved reputations. This was not Euler's way. On the contrary, once he had grasped a fruitful idea, he held on with an iron grip, squeezing out every last drop of information in an awesome exhibition of both genius and tenacity. So it was in this case.

For instance, he turned his attention to finding the exact sum of p-series with $p > 2$. Euler realized that this would require him to determine explicitly the coefficients of x^4, x^6, and so on in equation (3.4). Fortunately the tools necessary for such a determination were available in what are now called "Newton's formulas." These, published in Newton's *Arithmetica Universalis*,

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[8]Euler, *Introduction to Analysis of the Infinite*, Book I, pp. 154–155

[9]Johann Bernoulli, *Opera Omnia*, Vol 4, Georg Olms Verlagsbuchhandlung, Hildesheim, 1968 (Reprint), p. 22

[10]Weil, p. 184

describe the links between the roots and the coefficients of a polynomial. In Newton's words:

> ... the coefficient of the second term in an equation is, if its sign be changed, equal to the aggregate of all the roots under their proper signs; that of the third equal to the aggregate of the products of the separate roots two at a time; that of the fourth, if its sign be changed, equal to the aggregate of the products of the individual roots three at a time; that of the fifth equal to the aggregate of the products of the roots four at a time; and so on indefinitely.[11]

Here we shall give Euler's derivation of formulas—equivalent to Newton's—relating the roots and coefficients.[12] His proof, which dates from 1750, took a most unusual path to his desired end, unexpectedly introducing techniques of differential calculus to solve a problem in algebra. Yet, promised Euler,

> even if [this derivation] seem exceedingly remote, nevertheless it perfectly resolves the entire situation.

His argument is so delightful, so thoroughly "Eulerian," that it deserves our attention.

Theorem. *If the nth degree polynomial* $P(y) = y^n - Ay^{n-1} + By^{n-2} - Cy^{n-3} + \cdots \pm N$ *is factored as* $P(y) = (y - r_1)(y - r_2)\cdots(y - r_n)$, *then*

$$\sum_{k=1}^{n} r_k = A,$$

$$\sum_{k=1}^{n} r_k^2 = A\sum_{k=1}^{n} r_k - 2B,$$

$$\sum_{k=1}^{n} r_k^3 = A\sum_{k=1}^{n} r_k^2 - B\sum_{k=1}^{n} r_k + 3C,$$

$$\sum_{k=1}^{n} r_k^4 = A\sum_{k=1}^{n} r_k^3 - B\sum_{k=1}^{n} r_k^2 + C\sum_{k=1}^{n} r_k - 4D, \text{ and so on.}$$

Proof. Euler's objective was to connect the polynomial's coefficients A, B, $C, \ldots, N$ and its roots $r_1, r_2, \ldots, r_n$. His first step, somewhat surprisingly, was

[11] Whiteside, ed., *The Mathematical Papers of Isaac Newton*, Vol 5, p. 359

[12] Euler, *Opera Omnia*, Ser. 1, Vol 6, pp 20–25

to take logs:

$$\ln P(y) = \ln(y - r_1) + \ln(y - r_2) + \cdots + \ln(y - r_n).$$

The next step was more unanticipated—he differentiated both sides to get:

$$\frac{P'(y)}{P(y)} = \frac{1}{y - r_1} + \frac{1}{y - r_2} + \cdots + \frac{1}{y - r_n}. \tag{3.5}$$

As a final bit of analytic magic, Euler converted each fraction $1/(y - r_k)$ into its equivalent geometric series:

$$\begin{aligned}\frac{1}{y - r_k} &= \frac{1}{y}\left(\frac{1}{1 - (r_k/y)}\right) = \frac{1}{y}\left(1 + \frac{r_k}{y} + \frac{r_k^2}{y^2} + \cdots\right)\\ &= \frac{1}{y} + \frac{r_k}{y^2} + \frac{r_k^2}{y^3} + \frac{r_k^3}{y^4} + \cdots.\end{aligned}$$

Therefore by (3.5)

$$\begin{aligned}\frac{P'(y)}{P(y)} &= \frac{1}{y - r_1} + \frac{1}{y - r_2} + \cdots + \frac{1}{y - r_n}\\ &= \frac{n}{y} + \left[\sum_{k=1}^{n} r_k\right]\frac{1}{y^2} + \left[\sum_{k=1}^{n} r_k^2\right]\frac{1}{y^3} + \left[\sum_{k=1}^{n} r_k^3\right]\frac{1}{y^4} + \cdots.\end{aligned} \tag{3.6}$$

Note that this expresses $P'(y)/P(y)$ in terms of the *roots* of the original polynomial.

Because $P(y) = y^n - Ay^{n-1} + By^{n-2} - Cy^{n-3} + \cdots \pm N$, we have the obvious alternative

$$\frac{P'(y)}{P(y)} = \frac{ny^{n-1} - A(n-1)y^{n-2} + B(n-2)y^{n-3} - C(n-3)y^{n-4} + \cdots}{y^n - Ay^{n-1} + By^{n-2} - Cy^{n-3} + \cdots \pm N}, \tag{3.7}$$

framed in terms of the *coefficients* of the polynomial. Yet again, Euler had found different formulas for the same quantity, a ploy we have seen him use to good effect twice before in this chapter.

Equating the expressions from (3.6) and (3.7), he cross-multiplied to get:

$$\begin{aligned}ny^{n-1} &- A(n-1)y^{n-2} + B(n-2)y^{n-3} - C(n-3)y^{n-4} + \cdots\\ &= (y^n - Ay^{n-1} + By^{n-2} - Cy^{n-3} + \cdots \pm N)\end{aligned}$$

$$\times\left(\frac{n}{y}+\left[\sum_{k=1}^{n} r_k\right]\frac{1}{y^2}+\left[\sum_{k=1}^{n} r_k^2\right]\frac{1}{y^3}+\cdots\right)$$

$$= ny^{n-1}+\left(-nA+\sum_{k=1}^{n} r_k\right)y^{n-2}$$

$$+\left(nB-A\sum_{k=1}^{n} r_k+\sum_{k=1}^{n} r_k^2\right)y^{n-3}-\cdots.$$

Both sides of this equation begin with ny^{n-1}. Thereafter, we compare coefficients of like powers of y and solve to get the desired relationships. For example, equating the coefficients of y^{n-2} yields:

$$-A(n-1) = -nA+\sum_{k=1}^{n} r_k, \quad \text{and thus} \quad \sum_{k=1}^{n} r_k = A.$$

From the coefficients of y^{n-3}, we get:

$$B(n-2) = nB - A\sum_{k=1}^{n} r_k + \sum_{k=1}^{n} r_k^2, \quad \text{so that} \quad \sum_{k=1}^{n} r_k^2 = A\sum_{k=1}^{n} r_k - 2B.$$

Indeed, one can push this many terms deeper into the expansion (as Euler did) to find

$$\sum_{k=1}^{n} r_k^3 = A\sum_{k=1}^{n} r_k^2 - B\sum_{k=1}^{n} r_k + 3C$$

and

$$\sum_{k=1}^{n} r_k^4 = A\sum_{k=1}^{n} r_k^3 - B\sum_{k=1}^{n} r_k^2 + C\sum_{k=1}^{n} r_k - 4D,$$

and so on, with each new sum expressed in terms of previous ones. These are the promised relationships. Q.E.D.

Convinced? There surely are points here deserving closer attention. For instance, when considering $\ln(y - r_k)$, Euler implicitly assumed that $y > r_k$. When expanding

$$\frac{1}{y-r_k} = \frac{1}{y}+\frac{r_k}{y^2}+\frac{r_k^2}{y^3}+\frac{r_k^3}{y^4}+\cdots$$

as a geometric series, an unspoken convergence assumption was present. Such matters become problematic should one extend these rules to an "infinite polynomial," which is exactly what Euler did.

Still, it is impossible not to be struck again by Euler's brilliance in attacking an algebraic theorem about roots and coefficients by means of logarithms, derivatives, and geometric series—all tools from his analytic arsenal. His was an extremely agile mind.

What do these formulas have to do with summing p-series? To answer that question, we consider a polynomial containing only even powers of x and factored as follows:

$$1 - Ax^2 + Bx^4 - Cx^6 + \cdots \pm Nx^{2n} = (1 - r_1x^2)(1 - r_2x^2)\cdots(1 - r_nx^2). \tag{3.8}$$

Substitute $1/y$ for x^2:

$$\begin{aligned} 1 - A\left(\frac{1}{y}\right) + B\left(\frac{1}{y}\right)^2 - C\left(\frac{1}{y}\right)^3 + \cdots \pm N\left(\frac{1}{y}\right)^n \\ = \left(1 - r_1\frac{1}{y}\right)\left(1 - r_2\frac{1}{y}\right)\cdots\left(1 - r_n\frac{1}{y}\right). \end{aligned}$$

Then multiply both sides by y^n to get:

$$y^n - Ay^{n-1} + By^{n-2} - Cy^{n-3} + \cdots \pm N = (y - r_1)(y - r_2)\cdots(y - r_n).$$

This of course is precisely the case Euler considered above. Hence for (3.8) also we have the formulas

(a) $\sum_{k=1}^{n} r_k = A,$

(b) $\sum_{k=1}^{n} r_k^2 = A\sum_{k=1}^{n} r_k - 2B,$ and

(c) $\sum_{k=1}^{n} r_k^3 = A\sum_{k=1}^{n} r_k^2 - B\sum_{k=1}^{n} r_k + 3C.$

Euler assumed that these relationships between coefficients and roots remain valid *even if both are infinitely plentiful*—that is, when the sum runs from $k = 1$ to ∞. He returned to (3.3)

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots$$
$$=\left[1-\frac{x^2}{\pi^2}\right]\left[1-\frac{x^2}{4\pi^2}\right]\left[1-\frac{x^2}{9\pi^2}\right]\left[1-\frac{x^2}{16\pi^2}\right]\cdots,$$

which looks exactly like an infinite version of (3.8) with $A = 1/3!$, $B = 1/5!$, $C = 1/7!$, and $r_k = 1/k^2\pi^2$ for $k = 1, 2, \ldots$.

According to (a), $\sum_{k=1}^{\infty} 1/k^2\pi^2 = 1/3! = 1/6$ and so $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. This, of course, is Euler's "sensational" result derived above. But (b) and (c) yield entirely new information:

(b) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k^2\pi^2}\right)^2 = A\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2} - 2B = \left(\frac{1}{3!}\right)^2 - \frac{2}{5!} = \frac{1}{90},$

and so $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90};$

(c) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k^2\pi^2}\right)^3 = A\sum_{k=1}^{\infty}\left(\frac{1}{k^2\pi^2}\right)^2 - B\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2} + 3C$

$$= \left(\frac{1}{3!}\right)\left(\frac{1}{90}\right) - \left(\frac{1}{5!}\right)\left(\frac{1}{6}\right) + 3\left(\frac{1}{7!}\right) = \frac{1}{945},$$

and thus $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^6} = \frac{\pi^6}{945}.$

These are very strange. In his original paper Euler pushed further to evaluate p-series for $p = 8$, 10, and 12 . Later, in a 1744 publication, he gave exact sums for even values of p up to the colossal, if slightly ridiculous,[13]

$$\sum_{k=1}^{\infty}\frac{1}{k^{26}} = \frac{2^{24}}{27!}(76977927\pi^{26}) = \frac{1315862}{11094481976030578125}\pi^{26}.$$

Here Euler was answering questions no one had ever before *asked*. Better yet, his work contained the seeds for future research, including a link to what are now called the Bernoulli numbers and a hint of the Riemann zeta function that would prove so significant in the nineteenth century.[14] It was indeed an impressive display by a young mathematician aptly described by François Arago as "analysis incarnate."[15]

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[13]Euler, *Opera Omnia*, Ser. 1, Vol. 14, p. 185.

[14]Ayoub, pp. 1067–1086.

[15]Howard Eves, *An Introduction to the History of Mathematics*, 5th ed., Saunders, New York, 1983, p. 330

Epilogue

Here we shall address three topics related to the work of this chapter. First, we provide Euler's alternate solution of the Basel Problem. Second, we describe his *application* of the discoveries recounted above. And finally, we discuss a subsidiary challenge that has resisted the efforts of Euler and all who followed.

As noted, some of Euler's contemporaries, while accepting his answer to the Basel Problem, wondered about the validity of the argument that got him there. Daniel Bernoulli was especially concerned and wrote Euler in this regard.[16] In an attempt to silence such doubters, Euler devised another, quite different, proof that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. Although unlike the first, it is every bit as masterful.[17]

This argument requires three preliminary results, each of which falls well within the scope of a modern calculus course.

A. Prove the identity $\frac{1}{2}(\sin^{-1} x)^2 = \int_0^x \frac{\sin^{-1} t}{\sqrt{1-t^2}}\, dt$:

This follows immediately from the substitution $u = \sin^{-1} t$.

B. Find a series expansion for $\sin^{-1} x$:

Recalling that

$$\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_0^x (1-t^2)^{-1/2}\, dt,$$

we replace the expression under the integral by its binomial series and integrate termwise to get

$$\begin{aligned}\sin^{-1} x &= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2^2\cdot 2!}t^4 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}t^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}t^8 + \cdots\right) dt \\ &= t + \frac{1}{2}\times\frac{t^3}{3} + \frac{1\cdot 3}{2\cdot 4}\times\frac{t^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\times\frac{t^7}{7} \\ &\quad + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\times\frac{t^9}{9} + \cdots \Big|_0^x\end{aligned}$$

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[16]Euler, *Opera Omnia*, Ser. 1, Vol 14, p 141

[17]Ibid., pp 178–181

$$= x + \frac{1}{2} \times \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \times \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \times \frac{x^7}{7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \times \frac{x^9}{9} + \cdots .$$

C. Prove the relation $\int_0^1 \frac{t^{n+2}}{\sqrt{1-t^2}}\,dt = \frac{n+1}{n+2}\int_0^1 \frac{t^n}{\sqrt{1-t^2}}\,dt$ for $n \geq 1$:

For

$$J = \int_0^1 \frac{t^{n+2}}{\sqrt{1-t^2}}\,dt,$$

apply integration by parts with $u = t^{n+1}$ and $dv = (t/\sqrt{1-t^2})\,dt$ to get

$$J = \left(-t^{n+1}\sqrt{1-t^2}\right)\Big|_0^1 + (n+1)\int_0^1 t^n\sqrt{1-t^2}\,dt$$

$$= 0 + (n+1)\int_0^1 \frac{t^n(1-t^2)}{\sqrt{1-t^2}}\,dt = (n+1)\int_0^1 \frac{t^n}{\sqrt{1-t^2}}\,dt - (n+1)J.$$

Therefore

$$(n+2)J = (n+1)\int_0^1 \frac{t^n}{\sqrt{1-t^2}}\,dt,$$

and the result follows.

Fine. We now follow Euler in assembling these components to re-prove his formula. Simply let $x = 1$ in (A) to get:

$$\frac{\pi^2}{8} = \frac{1}{2}(\sin^{-1} 1)^2 = \int_0^1 \frac{\sin^{-1} t}{\sqrt{1-t^2}}\,dt.$$

Next, replace $\sin^{-1} t$ with its series expansion from (B) and integrate termwise:

$$\frac{\pi^2}{8} = \int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt + \frac{1}{2 \cdot 3}\int_0^1 \frac{t^3}{\sqrt{1-t^2}}\,dt + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}\int_0^1 \frac{t^5}{\sqrt{1-t^2}}\,dt + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}\int_0^1 \frac{t^7}{\sqrt{1-t^2}}\,dt + \cdots .$$

Knowing that

$$\int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt = 1,$$

we evaluate the other integrals using the recursion in (C):

$$\frac{\pi^2}{8} = 1 + \frac{1}{2\cdot 3}\left[\frac{2}{3}\right] + \frac{1\cdot 3}{2\cdot 4\cdot 5}\left[\frac{2}{3}\times\frac{4}{5}\right] + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left[\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\right] + \cdots$$
$$= 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots,$$

a summation involving only the odd squares.

From here Euler needed the following simple observation to reach his desired end.

Theorem. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$

Proof.

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] + \left[\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \cdots\right]$$
$$= \left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] + \frac{1}{4}\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots\right]$$
$$= \frac{\pi^2}{8} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}.$$

Thus

$$\frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{8},$$

and so

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{4}{3}\times\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$ Q.E.D.

There, before us, is the solution of the Basel Problem. This derivation, so different from the first, is the work of an analyst at the top of his powers—and one who seems to be enjoying himself immensely.

The epilogue's second objective is to show Euler applying his formulas to other, seemingly unrelated, matters. Indeed, he asserted that the "principal use" of these results "is in the calculation of logarithms."[18] Although this claim may sound far-fetched, he was happy to explain what he had in mind.

[18] Euler, *Introduction to Analysis of the Infinite*, Book I, p 158

Consider again the chapter's pivotal equation, labeled (3.3):

$$P(x) = \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\left[1 - \frac{x^2}{16\pi^2}\right]\cdots.$$

Recalling that $P(x) = (\sin x)/x$ for $x \neq 0$, we cross-multiply to get the infinite product

$$\sin x = x\left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\left[1 - \frac{x^2}{16\pi^2}\right]\cdots,$$

a result that holds even if $x = 0$.

When confronting this (or any) product, Euler seemed unable to resist taking logarithms. Such was the case here, and as usual it paid off:

$$\ln(\sin x) = \ln x + \ln\left(1 - \frac{x^2}{\pi^2}\right) + \ln\left(1 - \frac{x^2}{4\pi^2}\right) + \ln\left(1 - \frac{x^2}{9\pi^2}\right) + \cdots$$

which, for $x = \pi/n$, becomes

$$\ln\left(\sin\frac{\pi}{n}\right) = \ln\pi - \ln n + \ln\left(1 - \frac{1}{n^2}\right) + \ln\left(1 - \frac{1}{4n^2}\right) + \ln\left(1 - \frac{1}{9n^2}\right) + \cdots.$$

Perhaps the reader is by now sufficiently familiar with Euler's methods to anticipate that his next step was to introduce the series expansion of $\ln(1 - x)$ to get:

$$\begin{aligned}
\ln\left(\sin\frac{\pi}{n}\right) &= \ln\pi - \ln n + \left[-\frac{1}{n^2} - \frac{1}{2n^4} - \frac{1}{3n^6}\cdots\right] \\
&\quad + \left[-\frac{1}{4n^2} - \frac{1}{32n^4} - \frac{1}{192n^6}\cdots\right] \\
&\quad + \left[-\frac{1}{9n^2} - \frac{1}{162n^4} - \frac{1}{2187n^6}\cdots\right] + \cdots \\
&= \ln\pi - \ln n - \frac{1}{n^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right) \\
&\quad - \frac{1}{2n^4}\left(1 + \frac{1}{16} + \frac{1}{81} + \cdots\right) \\
&\quad - \frac{1}{3n^6}\left(1 + \frac{1}{64} + \frac{1}{729} + \cdots\right) - \cdots.
\end{aligned}$$

Remarkably, this formula contains *precisely* the p-series Euler had evaluated. It follows that

$$\ln\left(\sin\frac{\pi}{n}\right) = \ln\pi - \ln n - \frac{1}{n^2}\left(\frac{\pi^2}{6}\right) - \frac{1}{2n^4}\left(\frac{\pi^4}{90}\right) - \frac{1}{3n^6}\left(\frac{\pi^6}{945}\right) - \cdots.$$

What emerges is a rapidly converging series for $\ln(\sin \pi/n)$. To see it in action, choose $n = 7$ and approximate

$$\begin{aligned}\ln\left(\sin\frac{\pi}{7}\right) &= \ln\pi - \ln 7 - \frac{1}{49}\left(\frac{\pi^2}{6}\right) - \frac{1}{4802}\left(\frac{\pi^4}{90}\right) \\ &\quad - \frac{1}{352947}\left(\frac{\pi^6}{945}\right) - \cdots \\ &\approx -0.83498,\end{aligned}$$

which, with only five terms, is accurate to within ± 0.00000005.

Euler had found a way of computing logarithms of sines with great efficiency. More remarkably, he did so while short-cutting the numerical values of the sines themselves, as he himself observed when he wrote:

> [w]ith these formulas, we can find both the natural and the common logarithms of the sine and cosine of any angle, *even without knowing the sines and cosines*. [italics added][19]

In spite of such success, Euler got nowhere on a fundamental problem: to evaluate the p-series for odd values of p. Even the simplest of these,

$$\sum_{k=1}^{\infty}\frac{1}{k^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{216} + \frac{1}{343} + \cdots,$$

resisted explicit solution. Euler's original proof—as it emerged from equation (3.3)—was obviously geared toward even powers of x, and thus even values of p. Odd exponents slipped through his net.

Euler was keenly aware of the situation. The best he could do in his 1735 paper was to evaluate the loosely related series[20]

$$1 - \frac{1}{27} + \frac{1}{125} - \frac{1}{343} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^3} = \frac{\pi^3}{32}.$$

This was an intriguing answer. Unfortunately, it was to the wrong question.

[19]Ibid , p. 165

[20]Euler, *Opera Omnia*, Ser. 1, Vol 14, p 80

For guidance, Euler again turned to numerical approximations.[21] Because $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$ and $\sum_{k=1}^{\infty} 1/k^4 = \pi^4/90$, he naturally conjectured that $\sum_{k=1}^{\infty} 1/k^3 = \pi^3/m$ for some integer m falling between 6 and 90. With customary zeal, Euler calculated $\sum_{k=1}^{\infty} 1/k^3 \approx 1.202056903$ and, setting this equal to π^3/m, deduced that $m = 25.79435$—hardly a promising result.

At a later point, Euler conjectured that

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \alpha(\ln 2)^2 + \beta\frac{\pi^2}{6}\ln 2$$

for rational numbers α and β.[22] Intriguing though this was, it too led him nowhere.

So what do we know today about $\sum_{k=1}^{\infty} 1/k^3$? The answer is, "Disappointingly little." Progress over the centuries has been minimal. Indeed, only in 1978 did Roger Apéry manage to show that $\sum_{k=1}^{\infty} 1/k^3$ sums to an *irrational* number.[23] His was an ingenious answer to a difficult question. Yet the conclusion was both unsurprising and unsatisfying—unsurprising because the irrationality of this sum had been universally anticipated even if never proved; unsatisfying because one would have preferred an *exact* answer, not a broad classification like "irrational." It is as though we were looking for Captain Kidd's treasure and Apéry brilliantly demonstrated that it could be found somewhere in the Solar System. Mathematicians had wanted something a little more specific.

Worse, the irrationality of the series with $p = 3$ has as yet no counterpart for $p = 5$, $p = 7$, or any of the higher odd powers. For these, we are no further along than when Euler put down his pen over two centuries ago.

In this sense, even after 300 years, Jakob Bernoulli's problem is with us still. Faced with the mystery of the odd-valued p-series, one is tempted to throw up one's hands and reissue Jakob's challenge from 1689: "If anyone finds and communicates to us that which has thus far eluded our efforts, great will be our gratitude."

Then hope for a 21st century Euler.

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[21] Ibid., p. 440

[22] Euler, Opera Omnia, Ser 1, Vol 4, pp 143–144

[23] Alfred van der Poorten, "A Proof that Euler Missed," *The Mathematical Intelligencer*, Vol. 1, No. 4, 1978, pp. 195–203.