Middlebury College

Calculus II: Spring 2018
Extra Credit Problems

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Some thoughts and advice:

- You should expect to spend at least several hours on a single problem.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem? can you spot any patterns? have you thought about all possible cases?*
- To receive full credit you must present your solution to me on the blackboard in my office.
- You are allowed to work with each other on these problems. However, you must present your solution to me on your own.
- You **are not allowed** to use any additional resources. If you are concerned then please ask.
- Problems are graded as follows:
  - (*) = at most 1 point
  - (**) = at most 2 points
  - (***) = at most 3 points

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1. (**) In this problem you will determine a continued fraction expansion of the real number $\sqrt{2}$.
   (a) Let (a_n) be a sequence. Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$ then (a_n) is convergent and $\lim_{n\to\infty} a_n = L$.
   (b) Define the sequence (a_n) where
   $$a_1 = 1, \quad a_{n+1} = 1 + \frac{1}{1+a_n}, \; n = 1, 2, 3, \ldots$$
      i. Write down the first eight terms of (a_n).
      ii. Use part (a) to show that (a_n) is convergent and $\lim_{n\to\infty} = \sqrt{2}$. Deduce the **continued fraction expansion**
   $$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\cdots}}$$

2. (**) In this problem you will show the existence of *Euler's constant* γ. It is not know whether γ is rational or irrational.
   (a) Show that
   $$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n},$$
   where $\ln(x) = \int_1^x \frac{1}{t} dt$ is the natural logarithim function.
   (b) Define
   $$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n), \qquad n = 1, 2, 3, \ldots$$
   Show that the sequence (a_n) is decreasing and that $a_n \geq 0$, for each n. The limit $\gamma = \lim_{n\to\infty} a_n$ is known as *Euler's constant*, after Leonhard Euler (1707-1783).

3. (**) Let x be a real number having decimal expansion

$$x = a.a_1a_2a_3\cdots a_r\overline{b_1b_2\cdots b_k}$$

For example, we write

$$0.123\overline{456} = 0.123456456456456\cdots$$

(a) Show that there exists integers p, q so that $x = \frac{p}{q}$. *In particular, the contrapositive statement is: if x is irrational then x does not have a repeating decimal pattern.*

(b) (*Additional (not required)*) Show that if $x = \frac{p}{q}$ is rational then it has a repeating decimal expansion.

4. (**) Divide the plane into regions using straight lines. Prove that those regions can be coloured with two colours so that no two regions that share a boundary have the same colour.

5. (**) In this problem you will show that Euler's number

$$e = \exp(1) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2.71828\ldots$$

is irrational using a *proof by contradiction* argument. We will <u>assume</u> that e is, in fact, a rational number (i.e. a ratio of two integers) and derive a statement of absurdity. Hence, it must be the case that e is irrational (any real number must be either rational *or* irrational).

Assume that e is rational. This means that there are two natural numbers p and q so that

$$e = \frac{p}{q}$$

We will assume that p, q satisfy the condition that p is not a multiple of q.

(a) Explain why $q \neq 1$. Deduce that $q > 1$. (*Hint: what would have to be true of e if $q = 1$?*)

(b) Let s_m be the m^{th} partial sum of the series $1 + \sum_{n=1}^{\infty} \frac{1}{n!}$. Show that $q!s_q$ is an integer. Deduce that

$$q!(e - s_q)$$

is an integer.

We will now show, by a different argument, that $q!(e-s_q)$ is *not* an integer. This contradiction of what we've just shown implies that our assumption that e is rational must be a false assumption. Hence, e can't possibly be rational, so it must be an irrational number.

Observe that, in the argument that follows, we never make use of our assumption that $e = p/q$.

(c) Using the definition of e as the limit of a series, show that

$$e - s_q = \sum_{n=q+1}^{\infty} \frac{1}{n!} = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \ldots$$

(d) Deduce that

$$q!\,(e - s_q) = \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \ldots$$

(e) Let

$$a_n = \frac{1}{(q+1)(q+2)\cdots(q+n)}.$$

Show that

$$a_n \leq \frac{1}{(q+1)^n}, \qquad n = 1, 2, 3, \ldots$$

Deduce that

$$q!(e - s_q) \leq \sum_{n=1}^{\infty} \frac{1}{(q+1)^n} = \frac{1}{q} < 1.$$

(f) Recall from Problem A1 that $e > s_q$. Show that

$$0 < q!(e - s_q) < 1.$$

Conclude that $q!(e - s_q)$ can't possibly be an integer.

6. (***) In this problem you will prove the **Riemann Rearrangement Theorem**:

Let $\sum a_n$ be a conditionally convergent series, r a real number. Then, there is a rearrangement (b_n) of the sequence (a_n) so that the series $\sum b_n$ converges to r.

Given a series $\sum a_n$ we define the series $\sum p_n$ whose terms (p_n) are all the positive terms of the sequence (a_n), and a series $\sum q_n$ whose terms (q_n) are all the negative terms of the sequence (a_n). Specifically,

$$p_n = \frac{a_n + |a_n|}{2}, \qquad q_n = \frac{a_n - |a_n|}{2}.$$

Observe that, if $a_n > 0$ then $p_n = a_n$ and $q_n = 0$, and if $a_n < 0$ then $q_n = a_n$ and $p_n = 0$.

(a) Suppose that $\sum a_n$ is absolutely convergent. Show that both of the series $\sum p_n$ and $\sum q_n$ are convergent.

(b) Suppose that $\sum a_n$ is conditionally convergent. Show that one of the series $\sum p_n$ or $\sum q_n$ must be divergent. Deduce that the corresponding sequence of partial sums is unbounded.

(c) Suppose that $\sum a_n$ is conditionally convergent. Show that *both* $\sum p_n$ and $\sum q_n$ must have unbounded sequences of partial sums.

(d) Let r be a real number.

i. Show that there exists N such that $\sum_{n=1}^{N} p_n > r$. Define N_1 to be the least natural number such that

$$S_1 \stackrel{def}{=} \sum_{n=1}^{N_1} p_n > r.$$

ii. Show that there exists M such that $\sum_{n=1}^{N_1} p_n + \sum_{n=1}^{M} q_n < r$. Define M_1 to be the least natural number such that

$$T_1 \stackrel{def}{=} \sum_{n=1}^{N_1} p_n + \sum_{n=1}^{M_1} q_n < r.$$

iii. Similarly, let $N_2 > N_1$ be the least natural number such that

$$S_2 \stackrel{def}{=} \sum_{n=1}^{N_2} p_n + \sum_{n=1}^{M_1} q_n > r.$$

Explain why N_2 exists.

iv. Similarly, let $M_2 > M_1$ be the least natural number such that

$$T_2 \stackrel{def}{=} \sum_{n=1}^{N_2} p_n + \sum_{n=1}^{M_2} q_n < r.$$

Explain why M_2 exists.

v. Continuing in this way, show that you can obtain an increasing sequence of integers

$$N_1 < N_2 < N_3 < \cdots \qquad M_1 < M_2 < M_3 < \cdots$$

and sums

$$S_k \stackrel{def}{=} \sum_{n=1}^{N_k} p_n + \sum_{n=1}^{M_{k-1}} q_n, \quad \text{and} \quad T_k \stackrel{def}{=} \sum_{n=1}^{N_k} p_n + \sum_{n=1}^{M_k} q_n$$

satisfying

$$0 < S_k - r < p_{M_k}, \quad \text{and} \quad 0 < r - T_k < -q_{M_k}.$$

vi. Explain why the rearrangement

$$(b_n) = (p_1, \dots, p_{N_1}, q_1, \dots, q_{M_1}, p_{N_1+1}, \dots, p_{N_2}, q_{M_1+1}, \dots, q_{M_2}, \dots),$$

satisfies $\sum b_n = r$. Deduce Riemann's Rearrangement Theorem.

7. (**) $2n$ dots are placed around the outside of the circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

8. For $n \geq k \geq 0$, define the **binomial coefficient** to be

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We define $\binom{n}{k} = 0$ if $n < k$.

(a) (*) Show Pascal's identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

(b) (*) Using induction and Pascal's identity, show

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

(c) (**) Fix a natural number c. Using induction and Pascal's identity, show

$$\sum_{k=0}^{n} \binom{k}{c} = \binom{n+1}{c+1}$$

9. (**) Let $A_n = \{1, \dots, n\}$ be the collection of natural number $1, \dots, n$. For example, $A_3 = \{1, 2, 3\}$. For $n = 1, 2, 3, \dots$, define

$$c_n = \text{the number of subsets of } A_n \text{ having an even number of elements}$$

For example, A_3 contains the subsets $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$, (containing 2 elements) and the empty set $\varnothing$ (the subcollection containing 0 elements). Hence, $c_n = 4$.

Using induction show that $c_n = 2^{n-1}$, for every natural number n.