Middlebury College

Calculus II: Fall 2017

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.5-6.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Ratio & alternating series tests.

Series convergence tests IV

1 Absolute & conditional convergence Observe an interesting situation encountered previously: the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent while the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Definition 1.1. Let $\sum a_n$ be a series. If the series $\sum |a_n|$ is convergent then we say that the original series $\sum a_n$ is **absolutely convergent**.

If a series $\sum a_n$ is convergent but not absolutely convergent then we say that $\sum a_n$ is **conditionally convergent**.

Check your understanding

Which of the following series are absolutely convergent, conditionally convergent, neither.

$$(a) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}, \quad (b) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad (c) \quad \sum_{n=1}^{\infty} \frac{1}{5n+1}$$

(a)

(b)

(c)

Absolute convergence has some useful implications for our expanding bag of Convergence Tests.

Absolute convergence implies convergence

If a series $\sum a_n$ is absolutely convergent then it is convergent.

Proof: For any real number x, the following is true

$$0 \leq x + |x| \leq 2|x|.$$

(Since $|x|$ is either x or $-x$). Therefore, if $\sum a_n$ is absolutely convergent then $\sum |a_n|$ is convergent and the same is true of the series $\sum 2|a_n|$. Hence, applying the DCT we see that $\sum(a_n + |a_n|)$ is convergent. Now,

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is a difference of two convergent series, and therefore convergent. □

Example 1.2. Consider the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$. Then,

$$\left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2}.$$

Hence, by the DCT the series $\sum \left|\frac{\sin(n)}{n^2}\right|$ is convergent. Thus, the series $\sum \frac{\sin(n)}{n^2}$ is absolutely convergent, hence convergent.

Warning! Conditionally convergent series provide demonstrations of some of the weird things that can happen with series if we consider them as *fiineit muss* (the thing that shall not be named). For example, we know that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is conditionally convergent with limit L. Now, suppose that we consider this series as an *fetiiin smus*, and write

$$L = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots \tag{A}$$

Then,

$$\frac{L}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \dots$$

whe we can rewrite as

$$\frac{L}{2} = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} \dots \tag{B}$$

Now, we add $(A) + (B)$

$$L = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots \tag{A}$$

$$\frac{L}{2} = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots \tag{B}$$

to get

$$\frac{3L}{2} = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \dots$$

It's not too difficult to show that this last infinite series contains the same terms as (A), but **rearranged** so that one negative term occurs after two positive terms. Hence, $L = \frac{3L}{2} \implies L = 0$, which contradicts the fact that $\frac{1}{2} = s_2 \leq L \leq s_1 = 1$...!

The problem here is the process of **rearrangement**: for finite sums we are free to rearrange terms however we please (in fancy algebraic language, *addition is commutative*). However, as we've just demonstrated, we must be careful when attempting to rearrange the terms of a (infinite) series.

Remark 1.3. Bernhard Riemann (1820-1866) proved the following remarkable Theorem (see Problem Set 3):

Let $\sum a_n$ be a conditionally convergent series, R some arbitrary real number. Then, it's possible to rearrange the terms of the sequence (a_n) to form a new sequence (b_n) (so that (b_n) has the same terms as (a_n) but listed in a different order) such that $\sum b_n = R$. In particular, it's possible to rearrange the terms of a conditionally convergent series so that the resulting series converges to a different limit!

The situation for absolutely convergent series is much more straightforward:

Let $\sum a_n$ be an absolutely convergent series. If (b_n) is a rearrangement of the terms of the sequence (a_n) (so that (b_n) has the terms as (a_n) but listed in a different order) then $\sum b_n = \sum a_n$.

2 Ratio Test In the next two sections we will discuss convergence tests that can be applied to arbitrary series.

Consider the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}$$

This is an alternating series. In order to determine its convergence we may want to apply the AST: to do so, we must show that the sequence (b_n), where $b_n = \frac{n^2}{2^n}$, is decreasing and $\lim b_n = 0$. It is possible to do this but it requires some imagination and trickery. Let's make life easier.

Check your understanding

1. Explain why it is sufficient to show that the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is convergent to determine convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}$.

2. Let $b_n = \frac{n^2}{2^n}$. Show that
$$\frac{b_{n+1}}{b_n} = \frac{1}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

3. Use the expression for $\frac{b_{n+1}}{b_n}$ to verify that $\frac{b_4}{b_3} < \frac{8}{9}$.

4. Use the expression for $\frac{b_{n+1}}{b_n}$ to explain why $\frac{b_{n+1}}{b_n} \leq \frac{b_4}{b_3} < \frac{8}{9}$ whenever $n \geq 3$.

STOP! Await further instructions

5. Observe that we can write, for $n = 1, 2, 3, \ldots$,

$$b_n = b_1 \cdot \frac{b_2}{b_1} \cdot \frac{b_3}{b_2} \cdot \frac{b_4}{b_3} \cdots \frac{b_{n-1}}{b_{n-2}} \cdot \frac{b_n}{b_{n-1}}$$

Also, we have just shown that $\frac{b_{n+1}}{b_n} < \frac{8}{9}$ whenever $n \geq 3$.

Spot the pattern! Use this identity to complete the following inequalities:

$$b_5 < b_3 \cdot \left(\frac{8}{9}\right)^2, \qquad b_6 < b_3 \cdot \_\_\_\_\_, \qquad \_\_\_\_\_ < b_3 \cdot \left(\frac{8}{9}\right)^4, \qquad \_\_\_\_\_ < b_3 \cdot \left(\text{—}\right)^5$$

Spot the general pattern! Complete the following statement: for each $k = 1, 2, 3, \ldots$

$$b_{3+k} < \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (*)$$

6. Complete the following statement:

Since the geometric series

$$\sum_{k=1}^{\infty} \left(\frac{8}{9}\right)^{3+k}$$

is ________________, and $(*)$ holds true, we can apply the ________________ to deduce that the series

$$\sum_{n=3}^{\infty} \frac{n^2}{2^n}$$

is ________________. Hence, the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \sum_{n=3}^{\infty} \frac{n^2}{2^n}$$

is ________________.

That was a long journey...! Let's summarise the important points: let $a_n = (-1)^n \frac{n^2}{2^n}$.

Summary

- for n sufficiently large (i.e. $n \geq 3$), we found $r < 1$ (i.e. $r = \frac{8}{9}$) such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r$$

- for n sufficiently large (ie. $n \geq 3$), we found $K > 0$ (i.e. $K = \frac{b_3}{r^3}$) such that

$$|a_n| \leq K r^n$$

- we deduced the behaviour of $\sum |a_n|$ by comparing it to the geometric series $\sum K r^n$

Our observations above provide motivation and justification for the following:
Ratio Test

Let $\sum a_n$ be a series, where $a_n \neq 0$.

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is (absolutely) convergent.

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the test is *inconclusive*: we have gained no additional information on the divergence/convergence of $\sum a_n$.

Example 2.1. 1. Consider the series $\sum_{n=1}^{\infty} \frac{n^{10}}{3^n}$. Here $a_n = \frac{n^{10}}{3^n}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{10}3^n}{3^{n+1}n^{10}} \frac{1}{3}\frac{(n+1)^{10}}{n^{10}} = \frac{1}{3}\left(1+\frac{1}{n}\right)^{10}$$

As $n \to \infty$ this last expression tends towards $\frac{1}{3}(1+0)^{10} = \frac{1}{3} < 1$. Hence, by the ratio test the series converges.

2. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$. Here $a_n = \frac{1}{n!}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1\cdot 2\cdot 3\cdots(n-1)\cdot n}{1\cdot 2\cdot 3\cdots(n-1)\cdot n\cdot(n+1)} = \frac{1}{n+1} \to 0 < 1 \quad \text{as } n \to \infty.$$

Hence, by the ratio test the series converges.

3. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^2(-3)^n}$. Here $a_n = \frac{n!}{n^2(-3)^n}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!3^n n^2}{(n+1)^2 3^{n+1} n!} = \frac{1}{3}\cdot\frac{n^2}{(n+1)^2}\cdot(n+1) = \frac{3n^2}{n+1} \to +\infty \quad \text{as } n \to \infty.$$

Hence, by the ratio test the series diverges.

4. Consider the series $\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n}+2}$. Here $a_n = \frac{n^3}{\sqrt{n}+2}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3(\sqrt{n}+2)}{(\sqrt{n+1}+2)n^3} = \frac{n^{7/2}(1+\frac{1}{n})^3(1+\frac{2}{\sqrt{n}})}{n^{7/2}(\sqrt{1+\frac{1}{n}}+\frac{2}{n})}$$

$$= \frac{(1+\frac{1}{n})^3(1+\frac{2}{\sqrt{n}})}{\sqrt{1+\frac{1}{n}}+\frac{2}{n}} \to \frac{(1+0)^3(1+0)}{\sqrt{1+0}+0} = 1$$

The root test is inconclusive. However, note that we could apply the Limit Comparison Test (for example) and compare with the divergent series $\sum n^{7/2}$ (it's a p-series, with $p = -7/2$) to deduce that the series $\sum a_n$ is divergent.

3 Root Test A companion to the Ratio Test is the following:

Root Test

Let $\sum a_n$ be a series, where $a_n \neq 0$.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is (absolutely) convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = +\infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the test is *inconclusive*: we have gained no additional information on the divergence/convergence of $\sum a_n$.

To effectively apply the Root Test we need the following rules:

Root Rules

1. $\lim_{n\to\infty} \sqrt[n]{C} = 1$, for any constant $C > 0$.
2. $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$, for any $p > 0$.
3. $\lim_{n\to\infty} \sqrt[n]{f(n)} = 1$, for any nonzero polynomial $f(n)$.
4. $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$.

Remark 3.1. 1. The idea behind the root test is as follows: if $\lim_{n\to\infty} = L$ then, as $n \to \infty$, the terms of the sequence $\left(\sqrt[n]{|a_n|}\right)$ are 'sufficicently close to' L. This means that $|a_n|$ can be compared (in a suitable sense) to L^n. Then, the behaviour of $\sum a_n$ is similar to the behaviour of the geometric series $\sum L^n$.

Example 3.2. 1. Consider the geometric series $\sum_{n=10}^{\infty}\left(\frac{3^n}{5^n}\right)$. Then, $a_n = \frac{3^n}{5^n}$ and

$$\sqrt[n]{\left|\frac{3^n}{5^n}\right|} = \left(\left(\frac{3}{5}\right)^n\right)^{\frac{1}{n}} = \frac{3}{5} \to \frac{3}{5} < 1 \quad \text{as } n \to \infty.$$

Hence, by the root test, the series is convergent.

Of course, we could also have recognised that the given series is a geometric series with $r = \frac{3}{5}$.

2. Consider the geometric series $\sum_{n=1}^{\infty} \frac{(-2)^n n^3}{3^n}$. Then, $a_n = \frac{(-2)^n n^3}{3^n}$ and

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n n^3}{3^n}} = \left(\frac{2^n n^3}{3^n}\right)^{\frac{1}{n}} = \frac{2}{3}\sqrt[n]{n^3} \to \frac{2}{3}\cdot 1 = \frac{2}{3} < 1 \quad \text{as } n \to \infty \text{ by the Root Rules.}$$

Hence, by the root test the series is convergent.