Calculus II: Fall 2017

Contact: `gmelvin@middlebury.edu`

September 27 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.5-6.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Ratio & alternating series tests..

Series convergence tests III

1 Alternating series tests We have considered tests of convergence for series having *positive terms* e.g. Direct Comparison Tests and Limit Comparison Tests. Of course, this doesn't help us out when we are interested in determining convergence of series whose terms are not positive. Let's investigate the progress that we can make.

Definition 1.1. A series of the form $\sum(-1)^n b_n$, where $b_n \geq 0$ for all n, is called an **alternating series**.

Remark 1.2.
1. An alternating series is a series whose successive terms have alternating sign.
2. A *necessary condition* that an alternating series is convergent is that $\lim_{n\to\infty} b_n = 0$. However, this condition is *not sufficient*, as we will soon see.

Example 1.3. The following series are examples of alternating series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}, \qquad \sum_{n=3}^{\infty} \frac{\cos(n\pi)}{n^2+4n+4},$$

The following series are *not* alternating:

$$\sum_{n=1}^{\infty} \frac{2-(-1)^n}{n}, \qquad \sum_{n=4}^{\infty} \frac{\sin(n)}{n^2}.$$

We would like to determine conditions that the sequence (b_n) must satisfy so that the alternating series $\sum(-1)^n b_n$ is convergent. We already know that if $\sum(-1)^n b_n$ is convergent then it must be the case that $\lim_{n\to\infty} b_n = 0$. Do we require any further conditions?

Check your understanding

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

This is an alternating series with $b_n = \frac{1}{2n+1}$, $n = 1, 2, 3, \ldots$.

1. What adjectives would you use to describe the sequence (b_n)?

2. Write down the first six partial sums $s_1, s_2, s_3, s_4, s_5, s_6$. You do not need to simplify your expressions.

3. On the number line above plot a rough estimate of the values $s_1, s_2, s_3, s_4, s_5, s_6$. (*Hint: you should not need to compute the numerical value of any partial sum*)

4. Do you think the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ is convergent or divergent? If you think the series is convergent, indicate on the number line where its limit L will be; if you think the series is divergent, justify your conclusion.

5. Complete the following statement:

   Alternating Series Test (AST)

   > Let $\sum_n (-1)^n b_n$ be an alternating series, where $b_n > 0$ for $n = 1, 2, 3, \ldots$.
   > Suppose
   > $$(b_n) \text{ is } \underline{\hspace{4cm}}, \qquad \text{and} \qquad \lim_{n\to\infty} b_n = 0$$
   > Then, the series $\sum_n (-1)^n b_n$ is convergent.

   STOP! Await further instructions.

6. Using the number line, explain why $s_6 \le s_{10} \le s_5$. Can you provide an estimate of $s_{10^{10^{10}}}$?

7. Let $L = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$. Using the number line, explain why $|L - s_4| \le b_5$. (*Recall that* $|a-b|$ *is 'the distance between* a *and* b*'*). Determine the largest natural number k such that $|L - s_5| \le b_k$.

8. Complete the following statement:

Alternating Series Estimation Theorem (ASET)

Let $\sum_n (-1)^n b_n$ be an alternating series, where $b_n > 0$ for $n = 1, 2, 3, \ldots$. Suppose

$$(b_n) \text{ is } \underline{\hspace{4cm}}, \qquad \text{and} \qquad \lim_{n\to\infty} b_n = 0$$

and write $L = \sum_n (-1)^n b_n$. Then, $|L - s_n| \le$ ______.

Example 1.4. 1. Consider the alternating series given above:

$$(a) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n}, \qquad (b) \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}, \qquad (c) \quad \sum_{n=3}^{\infty} \frac{\cos(n\pi)}{n^2 + 4n + 4},$$

Then,

(a) convergent: $b_n = \frac{1}{n}$ is decreasing and $\lim_{n\to\infty} b_n = 0$. Hence, by AST the series is convergent.

(b) divergent: let $a_n = (-1)^{n-1}\frac{n}{n+1} = (-1)^{n-1}\frac{1}{1+\frac{1}{n}}$. Then, (a_n) is not convergent so that $\lim a_n$ does not exist. Hence, the series is divergent by the Divergence Test.

(c) convergent: observe that $\cos(n\pi) = (-1)^n$. This series is an alternating series with $b_n = \frac{1}{n^2+4n+4} = \frac{1}{(n+2)^2}$. As (b_n) is decreasing and $\lim_{n\to\infty} b_n = 0$ the series is convergent by the AST.

Remark 1.5. Consider the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n, \quad \text{where} \quad b_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd,} \\ \frac{1}{2^n}, & \text{if } n \text{ even.} \end{cases}$$

Observe that the first few terms of the sequence (b_n) are

$$1, \frac{1}{4}, \frac{1}{3}, \frac{1}{16}, \frac{1}{5}, \frac{1}{64}, \ldots$$

In particular, the sequence is not decreasing. The partial sums are

$s_1 = 1$

$s_2 = 1 - \frac{1}{4}$

$s_3 = 1 - \frac{1}{4} + \frac{1}{3}$

$s_4 = 1 - \frac{1}{4} + \frac{1}{3} - \frac{1}{16}$

$\vdots$

It can be shown that if $m = 2k$ is a very large even integer then

$$s_{2k} = 1 + \frac{1}{3} + \ldots + \frac{1}{2k-1} - \frac{1}{4} \cdot \frac{1 - (-\frac{1}{4})^k}{1 + \frac{1}{4}}$$

For k very large (hence m very large), it can be shown that

$$s_m = s_{2k} > \frac{1}{2} t_k - 1,$$

where (t_k) is the sequence of partial sums associated to the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$. As the sequence (t_k) is unbounded, the same is true of the sequence (s_m). Hence, the series $\sum (-1)^{n+1} b_n$ is divergent.

2 Absolute & conditional convergence Observe an interesting situation encountered above: the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent while the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Definition 2.1. Let $\sum a_n$ be a series. If the series $\sum |a_n|$ is convergent then we say that the original series $\sum a_n$ is **absolutely convergent**.

If a series $\sum a_n$ is convergent but not absolutely convergent then we say that $\sum a_n$ is **conditionally convergent**.

CHECK YOUR UNDERSTANDING

Which of the following series are absolutely convergent, conditionally convergent, neither.

$$(a) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}, \quad (b) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad (c) \quad \sum_{n=1}^{\infty} \frac{1}{5n+1}$$

(a)

(b)

(c)

Absolute convergence has some useful implications for our expanding bag of Convergence Tests.

Absolute convergence implies convergence

If a series $\sum a_n$ is absolutely convergent then it is convergent.

Proof: For any real number x, the following is true

$$0 \leq x + |x| \leq 2|x|.$$

(Since $|x|$ is either x or $-x$). Therefore, if $\sum a_n$ is absolutely convergent then $\sum |a_n|$ is convergent and the same is true of the series $\sum 2|a_n|$. Hence, applying the DCT we see that $\sum (a_n + |a_n|)$ is convergent. Now,

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is a difference of two convergent series, and therefore convergent. $\square$

Example 2.2. Consider the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$. Then,

$$\left| \frac{\sin(n)}{n^2} \right| \leq \frac{1}{n^2}.$$

Hence, by the DCT the series $\sum \left| \frac{\sin(n)}{n^2} \right|$ is convergent. Thus, the series $\sum \frac{\sin(n)}{n^2}$ is absolutely convergent, hence convergent.