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September 25 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.4.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Comparison tests, alternating series tests.

Series convergence tests II

1 Comparison tests In this paragraph we will be concerned with series $\sum a_n$ associated to sequences (a_n) **consisting of positive terms** i.e. for each $n = 1, 2, 3, \ldots$, we require $a_n > 0$.

Check your understanding

Consider the series $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$. Let (s_m) be the sequence of partial terms associated to this series. We are going to investigate the behaviour of this series by comparing it with the known behaviour of the (convergent) geometric series $\sum_{n=1}^{\infty} \frac{1}{5^n}$.

1. Write down the first three partial sums s_1, s_2, s_3. You do not need to simplify your expressions.

2. For each $m = 1, 2, 3, \ldots$, explain carefully why $s_{m+1} \geq s_m$. Deduce that (s_m) is an increasing sequence.

3. Recall the geometric series $\sum_{n=1}^{\infty} \frac{1}{5^n}$. Let (t_m) be the associated sequence of partial sums. We have seen that (t_m) is convergent with limit $\frac{1}{4}$. Complete the following statement:

   For each $n = 1, 2, 3, \ldots$, we have $5^n < 5^n + 3$ so that $\frac{1}{5^n+3} <$ ________.

   Hence, for each $m = 1, 2, 3, \ldots$, ______ $< s_m < t_m <$ ______

4. Using what you have discovered in the previous problems, explain carefully why (s_m) is convergent. Deduce that the series $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$ is convergent. Can you determine its limit?

Direct Comparison Test (DCT)

> Let $\sum a_n$ and $\sum b_n$ be series having positive terms.
>
> 1) Suppose that, for each n, $a_n \leq b_n$, and $\sum b_n$ is convergent. Then, $\sum a_n$ is convergent.
>
> 2) Suppose that, for each n, $a_n \geq b_n$, and $\sum b_n$ is divergent. Then, $\sum a_n$ is divergent.

The Direct Comparison Test has the following immediate consequences: first, in Problem Set 1, Problem B8, you show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent; second, we've seen that the Harmonic Series $\sum \frac{1}{n}$ is divergent.

***p*-series Test**

> Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number. Then,
>
> 1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p \geq 2$.
> 2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 1$.

Proof:

1. If $p \geq 2$ then, for each $n = 1, 2, 3, \ldots$,

$$n^p \geq n^2 \quad \Longrightarrow \quad \frac{1}{n^p} \leq \frac{1}{n^2}$$

   Hence, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can apply DCT to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also convergent.

2. If $p \leq 1$ then, for each $n = 1, 2, 3, \ldots$,

$$n^p \leq n \quad \Longrightarrow \quad \frac{1}{n} \leq \frac{1}{n^p}$$

   Hence, since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, we can apply DCT to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also divergent.

$\square$

Remark 1.1. We will soon see that the series $\sum \frac{1}{n^p}$ is convergent if $p > 1$ (i.e. we'll show convergence when $1 < p \leq 2$).

> The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is
>
> - convergent if $p > 1$.
> - divergent if $p \leq 1$.

Example 1.2.

Consider the series $\sum_{n=1}^{\infty} \frac{2}{4n^2+3n+1}$. As n gets large, the terms of this series will approximately 'look like' $\frac{2}{4n^2} = \frac{1}{2n^2}$. This observation motivates us to try to compare $\sum_{n=1}^{\infty} \frac{2}{4n^2+3n+1}$ with the convergent series $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$.

We note that, for $n = 1, 2, 3, \ldots$,

$$2n^2 < 2n^2 + \frac{3}{2}n + \frac{1}{2} \qquad \Longrightarrow \qquad \frac{2}{4n^2 + 3n^2 + 1} < \frac{1}{2n^2}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we apply the DCT to conclude that the series $\sum_{n=1}^{\infty} \frac{2}{4n^2+3n+1}$ is also convergent.

Sometimes we may have to be a little bit clever when trying to apply DCT.

Get creative!

Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$. As n gets large, the terms of this series will approximately 'look like' $\frac{1}{\sqrt{n}}$. This observation motivates us to try to compare $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ with the divergent series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

1. Explain why we can't use DCT to compare $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ with the divergent series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

2. For which natural numbers n is the inequality $n^2 > n + 2$ true?

3. Determine $0 < p \leq 1$ and k so that the following statement is true:

$$\text{if } n \geq k \text{ then } \frac{1}{n^p} \leq \frac{1}{\sqrt{n+2}}.$$

4. Apply the DCT to deduce that $\sum_{n=k}^{\infty} \frac{1}{\sqrt{n+2}}$ is divergent. Explain why this implies that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is divergent.

Trying to compare a given series with an appropriate candidate to apply the DCT is bit of an art. We'd like for life to be easier. Thankfully, it is.

Limit Comparison Test (LCT)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If the sequence $\left(\frac{a_n}{b_n}\right)$ is convergent and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge or both series diverge.

Remark 1.3. You will be given the opportunity to try and prove the Limit Comparison Test on Problem Set 2.

Example 1.4. 1. We will see how the LCT makes determining convergence/divergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ much easier.

   Indeed, since

$$\frac{\sqrt{n+2}}{\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n}}.\sqrt{1+\frac{2}{n}} = \sqrt{1+\frac{2}{n}} \to 1 \quad \text{as } n \to \infty,$$

   and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, we apply the LCT to conclude that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ also diverges.

2. Consider the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n}-2}$. As n gets large the terms of this series 'look like' $\frac{n}{\sqrt{n}} = \sqrt{n}$. We will use the LCT to compare $\sum \frac{n}{\sqrt{n}-2}$ with the divergent series $\sum \sqrt{n}$.

   Note that

$$\frac{n}{\sqrt{n}-1} \cdot \frac{1}{\sqrt{n}} = \frac{n}{n} \cdot \frac{1}{1-\frac{1}{\sqrt{n}}} = \frac{1}{1-\frac{1}{\sqrt{n}}} \to 1 \quad \text{as } n \to \infty.$$

   Since the series $\sum \sqrt{n}$ is divergent, we can apply the LCT to determine that the series $\sum \frac{n}{\sqrt{n}-1}$ is divergent also.

2 Alternating series

Definition 2.1. A series of the form $\sum(-1)^n b_n$, where $b_n \geq 0$ for all n, is called an **alternating series**.

Remark 2.2. An alternating series is a series whose successive terms have alternating sign.

Example 2.3. The following series are examples of alternating series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1}, \qquad \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n^2+4n+4}$$

The following series are *not* alternating:

$$\sum_{n=1}^{\infty} \frac{2-(-1)^n}{n}, \qquad \sum_{n=4}^{\infty} \frac{\sin(n)}{n^2}, \qquad \sum_{n=1}^{\infty} \left((-1)^n - \frac{(-1)^n}{2^n}\right).$$