Middlebury College

Calculus II: Fall 2017

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.2.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Infinite sequences, basic convergence tests.

Series convergence tests I

1 Telescoping series Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Check your understanding

1. Determine the first five partial sums s_1, s_2, s_3, s_4, s_5 as a fraction in simplest terms.

2. What do you expect to be the expression for s_m, the m^{th} partial sum?

3. Based on your guess above, is the sequence of partial sums (s_m) convergent or divergent? If convergent, what does this tell us about $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?; if divergent, give a careful justification.

Remark 1.1. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is an example of a *telescoping series*: the partial sums s_m can be shown to be a difference of two similar sums with successive cancellation.

2 Test for divergence In this paragraph we will discuss a simple test for divergence of a series $\sum_{n=1}^{\infty} a_n$. First we make the following observation: let (s_m) be the sequence of partial sums associated to the series $\sum_{n=1}^{\infty} a_n$. Then, we can recover the sequence (a_n) from the sequence of partial sums by noting that

$a_1 = s_1,$
$a_2 = (a_1 + a_2) - a_1 = s_2 - s_1,$
$a_3 = (a_1 + a_2 + a_3) - (a_1 + a_2) = s_3 - s_2,$
$a_4 = (a_1 + a_2 + a_3 + a_4) - (a_1 + a_2 + a_3) = s_4 - s_3,$
$\vdots$

Hence, for each $n = 1, 2, 3, \ldots$,

$$a_{n+1} = (a_1 + a_2 + \ldots + a_{n+1}) - (a_1 + \ldots + a_n) = s_{n+1} - s_n.$$

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES!

1. Let (b_n) be a convergent sequence, $\lim_{n\to\infty} b_n = L$. Define a new sequence

$$(c_n) = (b_2, b_3, b_4, \ldots),$$

so that $c_1 = b_2$, $c_2 = b_3$ etc. Complete the statement:

$$(c_n) \text{ is } \_\_\_\_\_\_\_\_\_\_\_\_ \text{ and } \lim_{n\to\infty} c_n = \_\_\_\_\_\_\_\_\_\_.$$

2. Let (s_m) be the sequence of partial sums associated to the series $\sum_{n=1}^{\infty} a_n$. Assume that $\sum_{n=1}^{\infty} a_n$ is convergent.

   (a) Using the previous exercise, explain carefully why $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$.

   (b) Complete the following statement:

   If the series $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n =$__________.

Considering the *contrapositive statement*[1] we obtain the following

Test for divergence

Let $\sum_{n=1}^{\infty} a_n$ be a series. If $\lim a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Remark 2.1. We will see in the next paragraph a series $\sum_{n=1}^{\infty} a_n$ that is *divergent* and for which $\lim_{n\to\infty} a_n = 0$. This shows that the converse of the above statement does not hold.

Example 2.2. Consider the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 1}{5n^2 + 6n + 1}.$$

This is the series associated to the sequence (a_n), where $a_n = \frac{2n^2+1}{5n^2+6n+1}$. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2 + 1}{5n^2 + 6n + 1} = \frac{2}{5} \neq 0,$$

the series $\sum_{n=1}^{\infty} \frac{2n^2+1}{5n^2+6n+1}$ does not converge, by the test for divergence.

[1]Given a statement of the form *if P then Q*, the *contrapositive statement* is the logically equivalent statement *if 'not Q' then 'not P'*. For example, the statement '*if you are a Vermonter then you are American*' is logically equivalent to '*if you are not an American then you are not a Vermonter*'.

3 The harmonic series

Definition 3.1. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the **harmonic series**.

We investigate the behaviour of the harmonic series. Denote the partial sums of the harmonic series by s_m, $m = 1, 2, 3, \ldots$.

SPOT THE PATTERN!

1. (a) Verify that

$$\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}, \quad \text{and} \quad \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

   (b) **Spot the pattern!**

$$\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} > \_\_\_\_ = \_\_\_\_$$

   (c) **Spot the general pattern!** Complete the following statement: for each $k = 1, 2, 3, \ldots$,

$$\tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \tfrac{1}{2^k+3} + \ldots + \tfrac{1}{2^{k+1}-1} + \tfrac{1}{2^{k+1}} > \_\_\_\_ = \_\_\_\_$$

   (d) Using (a),(b), explain why

$$s_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2},$$

$$s_{2^3} = 1 + \frac{1}{2} + \ldots + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$s_{2^4} = 1 + \frac{1}{2} + \ldots + \frac{1}{16} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}$$

   (e) **Spot the general pattern!** Complete the following statement: for each $k = 1, 2, 3, \ldots$,

$$s_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^{k+1}} > 1 + \_\_\_\_$$

   (f) Complete the following statement

The sequence of partial sums (s_m) associated to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is ____________.
Hence, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is ____________.

Appendix: telescoping sums

Note that we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Hence, the m^{th} partial sum of the series can be written as

$$s_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{m} \frac{1}{n} - \sum_{n=1}^{m} \frac{1}{n+1}$$

Expanding the sigma notation (this is always useful to fo!), we have

$$\begin{aligned} s_m &= \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}\right) - \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} + \frac{1}{m+1}\right) \\ &= 1 - \frac{1}{m+1}. \end{aligned}$$

Hence, the sequence of partial sums (s_m) is increasing, bounded (above by 1, below by 0), therefore convergent. As $\lim_{m\to\infty} \frac{1}{m+1} = 0$, we use the Limit Laws to obtain $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{m\to\infty} s_m = 1$.

Example 3.2. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)}.$$

We can write, for $n = 2, 3, 4, \ldots$,

$$\frac{1}{(n-1)(n+1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

Hence, the m^{th} partial sum is

$$\begin{aligned} s_m &= \sum_{n=2}^{m} \frac{1}{(n-1)(n+1)} \\ &= \sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\ &= \frac{1}{2}\left(\sum_{n=2}^{m} \frac{1}{n-1} - \sum_{n=2}^{m} \frac{1}{n+1}\right) \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{m-1}} - \cancel{\frac{1}{3}} - \ldots - - \cancel{\frac{1}{m-1}} - \frac{1}{m} - \frac{1}{m+1}\right) \\ &= \frac{1}{2}\left(\frac{3}{2} - \frac{1}{m} - \frac{1}{m+1}\right) \end{aligned}$$

Hence, as $\lim_{m\to\infty} \frac{1}{m} = \lim_{m\to\infty} \frac{1}{m+1} = 0$, the sequence (s_m) is convergent with limit $\frac{3}{4}$. Therefore, the series is convergent and $\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} = \frac{3}{4}$.