Contact: gmelvin@middlebury.edu

September 21 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1-2.
- *Calculus*, Spivak, 3rd Ed.: Section 22, 23.
- *AP Calculus BC*, Khan Academy: Infinite sequences, finite geometric series, partial sums, infinite geometric series.

End of sequences. Beginning of series.

1 Monotonic+Bounded Theorem

Theorem 1.1. *Let (a_n) be a monotonic, bounded sequence. Then, (a_n) is convergent.*

Remark 1.2.

1. The Monotonic+Bounded Theorem is a little strange: it tells us that a monotonic, bounded sequence is convergent but does not say say how to find $\lim_{n\to\infty} a_n$! Compare this with the Squeeze Theorem where we not only show that a sequence is convergent but also obtain its limit.

2. In Problem Set 1 there is a generalisation of the Monotonic+Bounded Theorem: say that a sequence (a_n) is **eventually monotonic** if there is some n_0 such that the sequence $(a_n)_{n\geq n_0}$ is monotonic. For example, if $a_n = n^2 - 13n + 30$ then the sequence (a_n) is eventually monotonic (it is eventually increasing: the sequence $(a_n)_{n\geq 7}$ is increasing. Plot its graph for $n = 1, \dots, 10$ to see).

Example 1.3. Suppose that $0 \leq x < 1$. Consider the sequence (a_n), where $a_n = x^n$. Then, for each $n = 1, 2, 3, \dots$

$$a_{n+1} - a_n = x^{n+1} - x^n = x^n(x-1) \leq 0 \quad \Longrightarrow \quad a_{n+1} \leq a_n, \quad n = 1, 2, 3, \dots.$$

Hence, (a_n) is decreasing. Also, (a_n) is bounded: for each $n = 1, 2, 3, \dots$, we have $0 \leq a_n \leq 1$. Therefore, by the Monotonic+Bounded Theorem the sequence (a_n) is convergent. Let $L = \lim_{n\to\infty} a_n$.

Define a new sequence $(b_n) = (a_2, a_3, a_4, \dots)$, so that the n^{th} term of (b_n) is the $(n+1)^{st}$ term of (a_n). Then, (b_n) is also convergent with limit L (why?). Notice that, for $n = 1, 2, 3, \dots$,

$$b_n = a_{n+1} = x^{n+1} = xa_n.$$

Hence, using the Limit Laws for sequences we have

$$L = \lim_{n\to\infty} b_n = \lim_{n\to\infty}(xa_n) = x\left(\lim_{n\to\infty} a_n\right) = xL \quad \Longrightarrow \quad L(x-1) = 0$$

The last equality implies that $L = 0$, since we have assumed that $x < 1$.

Check your understanding

1. Let $0 \leq x < 1$. Consider the sequence (c_n), where $c_n = -x^n$. Explain carefully why $\lim_{n\to\infty} c_n = 0$.

2. Let $-1 < x < 1$. Consider the sequence (d_n), where $d_n = x^n$. Show that $\lim_{n\to\infty} d_n = 0$. (*Hint: denote $r = |x|$. Then, $-r^n \leq d_n \leq r^n$, for $n = 1, 2, 3, \ldots$. Now, squeeeeeze.*)

3. Let x be a real number such that $|x| > 1$ and define the sequence (e_n), where $e_n = x^n$. Complete the following statement:

   '*The sequence (e_n) is ______________. Hence, (e_n) is not ______________*'

2 Introduction to series Every real number x has a decimal expansion and this decimal expansion can have finite or infinite length. For example, we learn at some point that

$$\frac{1}{3} = 0.3333333\ldots$$

What does the right hand side of this equality mean? Why is the above equality true?

One way to rewrite this is as follows:

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots$$

Again, you might, and should, ask: *what does this mean?* In particular, *what does it mean to 'sum' an infinite number of terms*? This obviously(?) does not make any sense if we consider the sum

$$1 + 2 + 3 + 4 + \ldots = \quad ???$$

First, we have to note the following basic observation: **it is impossible to 'sum' an infinite number of terms** - there is (literally) not enough time to do so. However, it is possible to ask whether the sequence of finite sums

$$\begin{aligned} s_1 &= \tfrac{3}{10}, \\ s_2 &= \tfrac{3}{10} + \tfrac{3}{10^2}, \\ s_3 &= \tfrac{3}{10} + \tfrac{3}{10^2} + \tfrac{3}{10^3}, \\ &\vdots \\ s_m &= \tfrac{3}{10} + \tfrac{3}{10^2} + \ldots + \tfrac{3}{10^m} \end{aligned}$$

converges to a limit L. In *sigma notation*[1] we have

$$s_m = \sum_{n=1}^{m} \frac{3}{10^n}.$$

We give the following essential definition:

[1]See handout for basic properties of sigma notation.

Definition 2.1. Let (a_n) be a sequence.

1. Define **the m^{th} partial sum associated to** (a_n) to be the (finite) sum
$$s_m = a_1 + a_2 + \ldots + a_m = \sum_{n=1}^{m} a_n.$$
2. Define the **sequence of partial sums associated to** (a_n) to be the corresponding sequence (s_m), where s_m is the m^{th} partial sum associated to (a_n).
3. If (s_m) is convergent then we write
$$\sum_{n=1}^{\infty} a_n \overset{def}{=} \lim_{m\to\infty} s_m.$$
We call the symbol $\sum_{n=1}^{\infty} a_n$ a **series**.

Important Remark

1. A series is the *limit* of a sequence of finite sums.
2. Given a sequence (a_n) we will (by abuse of notation) define the symbol $\sum_{n=1}^{\infty} a_n$ to be a **series**, even when we don't know whether (s_m), the sequence of partial sums associated to (a_n), is convergent. We will say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent** if the sequence of partial sums associated to (a_n) is convergent, and **divergent** otherwise.
3. Given a sequence (a_n) we will call the series $\sum_{n=1}^{\infty} a_n$ **the series associated to** (a_n).
4. Let $\sum_{n=1}^{\infty} a_n$ be a series. We will also say that the partial sums associated to (a_n) are the **partial sums associated to** $\sum_{n=1}^{\infty} a_n$.
5. We have
$$\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} \sum_{n=1}^{m} a_n.$$

For the next few lectures we will be interested in determining when series are convergent and how we can use series to provide approximations to real numbers.

Check your understanding

1. Let (a_n) be a sequence such that, for each $m = 1, 2, 3, \ldots$, the m^{th} partial sum s_m satisfies
$$s_m = a_1 + a_2 + \ldots + a_m = \frac{2m-1}{3m+5}.$$
Does the series $\sum_{n=1}^{\infty} a_n$ converge? If so, what is its limit? If not, explain carefully why not.

2. Let (a_n) be a sequence such that the sequence of partial sums associated to (a_n), (s_m), satisfies
$$s_m = 10 - \frac{4}{m^2+1}.$$
Then, $\sum_{n=1}^{\infty} a_n =$ ________________.

Since a convergent series is, by definition, the limit of a sequence, we can translate the Limit Laws for Sequences into corresponding results for series.

Proposition 2.2 (Additive properties of series)**.** *Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be convergent series, c a constant. Then,*

1. $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$,
2. $\sum_{n=1}^{\infty} c a_n = c \left(\sum_{n=1}^{\infty} a_n\right)$

3 Geometric series

Definition 3.1. Let r be a real number. Define the sequence (a_n), where $a_n = r^n$. The series $\sum_{n=1}^{\infty} a_n$ is called a **geometric series**.

Check your understanding

1. Let $\sum_{n=1}^{\infty} r^n$ be a geometric series. Write down the m^{th} partial sum s_m associated to this series.

2. Show that
$$s_m - r s_m = r(1 - r^m).$$
Deduce that, if $r \neq 1$ then $s_m = \frac{r(1-r^m)}{1-r}$.

3. For which r is the sequence of partial sums (s_m) convergent? (*Hint: consider the exercise at the beginning of this lecture.*)

4. What happens when $r = \pm 1$?

Create your own Theorem!

Geometric Series Theorem

Let _____ $< r <$ _____. Then, the geometric series $\sum_{n=1}^{\infty} r^n$ is ______________ and

$$\sum_{n=1}^{\infty} r^n = \text{______________}.$$

Remark 3.2. A series $\sum_{n=1}^{\infty} a_n$ is a geometric series if there is a real number r so that, for each $n = 1, 2, 3, \ldots$,

$$\frac{a_{n+1}}{a_n} = r.$$

Example 3.3. 1. Consider the series

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$$

Using the Additive Properties for Series (Proposition 2.2), we have

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = 3 \sum_{n=1}^{\infty} \frac{1}{10^n}.$$

The series on the right hand side is a geometric series with $r = \frac{1}{10}$. Hence, since $|r| < 1$, we have

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = 3\left(\frac{1}{10} . \frac{1}{1 - \frac{1}{10}}\right) = \frac{1}{3}$$

2. The series $\sum_{n=1}^{\infty} (-2)^n 3^{2-n}$ is convergent: indeed, this is the series associated to the sequence (a_n), where

$$a_n = (-2)^n 3^{2-n} = 9\left(\frac{-2}{3}\right)^n.$$

Hence,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 9\left(\frac{-2}{3}\right)^n = 9\sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n$$

and we identify this latter series as a geometric series with $r = \frac{-2}{3}$. As $|r| < 1$, the series is convergent with limit

$$9\sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n = 9\left(\frac{-2}{3}\right)\frac{1}{1 + \frac{2}{3}} = -\frac{18}{5}.$$

Alternatively, we can identify the given series as a geometric series once we observe that, for each $n = 1, 2, 3, \ldots$,

$$\frac{a_{n+1}}{a_n} = \frac{(-2)^{n+1} 3^{2-n-1}}{(-2)^n 3^{2-n}} = \frac{-2}{3}.$$

An e-xcellent example

We are going to model compound interest and define the fundamental mathematical constant e as the limit of a sequence. We will need the results of our investigation into geometric series along the way.

Let

$$\begin{aligned} P &= \text{ amount in savings account at time } t = 0, \\ r &= \text{ annual interest rate, } \quad 0 \le r \le 1, \\ f(t) &= \text{ savings account balance at time } t \text{ (in years).} \end{aligned}$$

Thus, if we compound interest once at the end of each year then

$$\begin{aligned} f(0) &= P \\ f(1) &= P + rP = P(1+r) \\ f(2) &= (P + rP) + r(P + rP) = P(1+r)^2 \\ f(t) &= P(1+r)^t, \quad \text{where } t \text{ is a natural number.} \end{aligned}$$

If we compound interest m times throughout a single time interval then

$$\begin{aligned} f(0) &= P \\ f(1) &= P\left(1 + \tfrac{r}{m}\right)^m \\ f(2) &= P\left(1 + \tfrac{r}{m}\right)^{2m} \\ f(t) &= P\left(1 + \tfrac{r}{m}\right)^{tm}, \quad \text{where } t \text{ is a natural number.} \end{aligned}$$

Consider the situation when $P = r = 1$. We are interested in determining *instantaneous* compound interest: suppose the bank continuously compounds interest on our savings account, how much savings do we expect to have at the end of the first time interval. Mathematically, this means we want to determine the behaviour of $f(1) = \left(1 + \frac{1}{m}\right)^m$ as $m \to \infty$.

Consider the sequence (a_n), where $a_n = \left(1 + \frac{1}{n}\right)^n$. We compute some of the terms of the sequence

$$\begin{aligned} a_1 &= (1+1) = 2, \\ a_2 &= \left(1 + \tfrac{1}{2}\right)^2 = \tfrac{9}{4} = 2.25, \\ a_3 &= \left(1 + \tfrac{1}{3}\right)^3 = \tfrac{64}{27} = 2.37\ldots, \\ &\vdots \\ a_{1000} &= \left(1 + \tfrac{1}{1000}\right)^{1000} = 2.7169\ldots \end{aligned}$$

It appears that (a_n) is increasing: indeed, by the Binomial Theorem[2] we have

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + n.\frac{1}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \ldots + \frac{n.(n-1)\ldots 2.1}{1.2.\ldots .n}\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{1.2}\left(1 - \frac{1}{n}\right) + \frac{1}{1.2.3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots \\ &\quad + \frac{1}{1.2\ldots n}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \end{aligned}$$

[2]Recall, the Binomial Theorem states that

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \ldots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{1}xy^{n-1} + y^n,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Each of the expressions

$$\left(1-\frac{1}{n}\right),\ \left(1-\frac{2}{n}\right),\ \ldots,\ \left(1-\frac{n-1}{n}\right)$$

is increasing with respect to n, so that the same is true for any product of these expressions. Hence, $\left(1+\frac{1}{n}\right)^n$ also increases with n.

Next, we observe that, for each $n = 1, 2, 3, \ldots$,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+1+\frac{1}{1.2}\left(1-\frac{1}{n}\right)+\frac{1}{1.2.3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots \\ &\quad+\frac{1}{1.2\ldots n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\ &< 1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots+\frac{1}{1.2.3\ldots n} \\ &\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\end{aligned}$$

We identify this list expression as the sum $2+s_n$, where s_n is the n^{th} partial sum of the geometric series $\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}.\frac{1}{1-\frac{1}{2}}=1$. Hence,

$$\left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n} < 1+1+1=3.$$

In conclusion, the sequence (a_n), where $a_n=\left(1+\frac{1}{n}\right)n$, is increasing and bounded, and therefore convergent. Define

$$e \overset{def}{=} \lim_{n\to\infty} a_n.$$

As $2 \leq a_n < 3$, for every $n = 1, 2, 3, \ldots$, we must have $2 \leq e < 3$.

Remark 3.4. The number e just defined as the limit of the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\geq 1}$ is the base of the natural logarithm function. We will see later in the course that

$$e = 1+\sum_{n=1}\infty\frac{1}{n!} = 1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots$$