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September 20 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

Sequences, the tools

1 The Squeeze Theorem

Definition 1.1. Let n be a natural number. We define the natural number $n!$, said *n factorial*, to be the product

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

By definition, we set $0! \overset{def}{=} 1$.

Example 1.2. We have $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

Consider the sequence (a_n), where

$$a_n = \frac{n!}{n^n}, \quad n = 1, 2, 3, \ldots.$$

Check your understanding

1. Write down the first five terms of the sequence (a_n).

2. Do you think the sequence is increasing, decreasing, bounded?

3. Do you think the sequence is convergent/divergent? If convergent, what do you expect the limit to be? If divergent, can you explain why?

Draw graph

Spot the pattern Consider the sequence (c_n), where $c_n = \frac{1}{n}$, and plot its graph above (along with the graph of (a_n)).

1. What is the relation between the first five terms of (a_n) and (c_n)?

2. What do you expect to be the relation between the n^{th} terms a_n and c_n?

3. Using the sequence (c_n), carefully explain, either, why (a_n) is convergent and what its limit is, or, why (a_n) is divergent.

4. Let (b_n) be the sequence $b_n = 0$, for each $n = 1, 2, 3, \ldots$. What is the relation between the n^{th} terms of a_n and b_n?

Create your own Theorem!

The Squeeze Theorem

Let (a_n), (b_n) (c_n) be sequences, where (b_n) and (c_n) are convergent. Let $L = \lim_{n\to\infty} b_n$ and $L' = \lim_{n\to\infty} c_n$ satisfy the relation __________.

Furthermore, assume that, for each $n = 1, 2, 3, \ldots,$, the n^{th} terms a_n, b_n, c_n, satisfy the relation _______________.

Then, the sequence (a_n) is _______________ and $\lim_{n\to\infty} a_n =$ __________

Example 1.3. 1. Consider the sequence (a_n), where $a_n = \sin(n)/n$. Define the sequences (b_n), (c_n), where

$$b_n = -\frac{1}{n}, \qquad c_n = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$

Then, (b_n) and (c_n) are convergent and

$$\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} c_n$$

Moreover, for each $n = 1, 2, 3, \ldots$

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}.$$

Hence, by the Squeeze Theorem, (a_n) is convergent and $\lim_{n\to\infty} a_n = 0$.

2. We provide a rigorous justification that $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$, for each $n = 1, 2, 3, \ldots$. Of course, we have $\frac{n!}{n^n} \geq 0$, for each $n = 1, 2, 3, \ldots$. Now, for every $n = 1, 2, 3, \ldots,$

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}{n \cdot n \cdot n \cdots n \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right) \leq \frac{1}{n},$$

because the term in parentheses is ≤ 1.

Remark 1.4. We have the following (more general) modification of The Squeeze Theorem, which we will call the **Ultimate Squeeze Theorem**:

> Let (a_n), (b_n) (c_n) be sequences, where (b_n) and (c_n) are convergent. Suppose
> $$\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n.$$
> Furthermore, assume that, there is some natural number $n_0 \geq 1$ so that
> $$\text{if } n \geq n_0 \text{ then } b_n \leq a_n \leq c_n.$$
> Then, the sequence (a_n) is convergent and $\lim_{n\to\infty} a_n = L$

2 The Monotonic+Bounded Theorem We have seen several results that allow us to deduce when a sequence is convergent (e.g. Limit Laws, Squeeze Theorem). However, we don't yet have any tools to show that a sequence is *divergent* (i.e. *not convergent*).

> **Theorem 2.1** (Convergent implies bounded)**.** *Let (a_n) be a convergent sequence. Then, (a_n) is bounded.*

Proof: Write $L = \lim_{n\to\infty} a_n$. Let $\varepsilon = 1$. Then, we must be able to find some N such that[1]

$$n \geq N \quad \Longrightarrow \quad |a_n - L| < \varepsilon = 1 \quad \Longrightarrow \quad L - 1 < a_n < L + 1.$$

In particular, for all $n \geq N$, we have $|a_n| < |L| + 1$.

Write K for the largest of the numbers $|a_1|, \dots, |a_N|, |L| + 1$. Then, for this choice of K we have $|a_n| \leq K$, for each $n = 1, 2, 3, \dots$. That is, $-K \leq a_n \leq K$, for each $n = 1, 2, 3, \dots$, so that (a_n) is bounded. □

We will discuss a partial converse[2] to the 'Convergent implies Bounded' result just given. But first:

GET CREATIVE!

Give three examples of sequences (a_n), (b_n), (c_n) that are bounded but not convergent.

Definition 2.2. Let (a_n) be a sequence that is *either* increasing *or* decreasing (or both!). Then, we say that (a_n) is **monotonic**.

[1] Recall that the collection of symbols $|a_n - L| < \varepsilon$ means '*the distance from a_n to L is less than ε*'. This is equivalent to the pair of inequalities $L - \varepsilon < a_n < L + \varepsilon$.

[2] Given a statement '*if P then Q*', the **converse** is the statement '*if Q then P*'. Very important note: in general, the truth of a statement '*if P then Q*' does not determine the truth of the statement '*if Q then P*'. For example, compare the statements '*if you are a Vermonter then you are American*' and '*if you are American then you are a Vermonter*'.

Check your understanding

1. Draw the graphs of three (different) monotonic, bounded sequences (a_n), (b_n), (c_n).

2. What common features do the sequences (a_n), (b_n), (c_n) possess?

Create your own Theorem!

Monotonic+Bounded Theorem

Let (a_n) be a monotonic and bounded sequence. Then, (a_n) is ______________.

Remark 2.3.
1. The Monotonic+Bounded Theorem is a little strange: it tells us that a monotonic, bounded sequence is convergent but does not say say how to find $\lim_{n\to\infty} a_n$! Compare this with the Squeeze Theorem where we not only show that a sequence is convergent but also obtain its limit.

2. In Problem Set 1 there is a generalisation of the Monotonic+Bounded Theorem: say that a sequence (a_n) is **eventually monotonic** if there is some n_0 such that the sequence $(a_n)_{n\geq n_0}$ is monotonic. For example, if $a_n = n^2 - 13n + 30$ then the sequence (a_n) is eventually monotonic (it is eventually increasing: the sequence $(a_n)_{n\geq 7}$ is increasing).

Example 2.4. Suppose that $0 \leq x < 1$. Consider the sequence (a_n), where $a_n = x^n$. Then, for each $n = 1, 2, 3, \ldots$

$$a_{n+1} - a_n = x^{n+1} - x^n = x^n(x-1) \leq 0 \quad \Longrightarrow \quad a_{n+1} \leq a_n, \quad n = 1, 2, 3, \ldots.$$

Hence, (a_n) is decreasing. Also, (a_n) is bounded: for each $n = 1, 2, 3, \ldots$, we have $0 \leq a_n \leq 1$. Therefore, by the Monotonic+Bounded Theorem the sequence (a_n) is convergent.