Middlebury College

Calculus II: Fall 2017

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

Sequences, first steps

1 Recap

Definition 1.1. Let (a_n) be a sequence. We say that (a_n) is a **convergent sequence with limit** L if, for any $\epsilon > 0$, there is some natural number N such that

$$n \geq N \quad \Longrightarrow \quad |a_n - L| < \varepsilon.$$

If (a_n) is *not convergent* then we say that (a_n) is **divergent**.

Remark 1.2.

1. Recall that, for real numbers x, y, the non-negative real number $|x-y|$ is the (unsigned) distance between x and y. Thus, the mathematical definition given above is to be read as '*(a_n) is convergent with limit L if, for any $\varepsilon > 0$, the property $D_{\varepsilon,L}$ (see September 15 lecture) holds for (a_n) as $n \to \infty$.*'
2. If (a_n) is convergent with limit L then we write
$$\lim_{n\to\infty} a_n = L, \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty.$$
3. Suppose that $f(x)$ is a function defined for all $1 \leq x \leq \infty$. If (a_n) is a sequence so that $a_n = f(n)$, for $n = 1, 2, 3, \ldots$, then $\lim_{n\to\infty} a_n = L$ precisely whenever $\lim_{x\to\infty} f(x) = L$.
4. In this class, the adjective *divergent* is synonymous with *not convergent*.

Example 1.3. Consider the sequence (a_n), where $a_n = \frac{1}{n}$. We will show directly that (a_n) is convergent with limit $L = 0$.

Suppose we are given some fixed $\varepsilon > 0$. To verify that (a_n) satisfies the statement of Definition 1.1 we have to find an N such that, for each $n \geq$N, we necessarily have

$$|a_n - 0| = \left|\frac{1}{n}\right| < \varepsilon.$$

Observe that, since $a_n > 0$ for all $n = 1, 2, 3, \ldots$, we have $|a_n| = a_n$. Hence, we need N so that, for each $n \geq N$, we necessarily have

$$\frac{1}{n} = a_n = |a_n| < \varepsilon.$$

Rearranging the above inequality, if we take a natural number $N > \frac{1}{\varepsilon}$ then

$$n \geq N \quad \Longrightarrow \quad n \geq N > \frac{1}{\varepsilon} \quad \Longrightarrow \quad |a_n| = \frac{1}{n} < \varepsilon.$$

Thus, we have shown *directly* that $\lim_{n\to\infty} \frac{1}{n} = 0$.

2 Limit laws for sequences Showing directly that a sequence (a_n) is convergent with limit L (as in the Example above) can be difficult! For example, it (hopefully) seems intuitive that $\lim_{n\to\infty} \frac{3n+1}{n^2+10} = 0$ (the denominator will dominate the numerator as n gets very large), but finding N so that our definition holds looks like a formidable task. This is why mathematical Theorems/Propositions are helpful - they provide us with power tools that can make life easier.

Let (a_n) be a sequence and consider its graph. Then, it's always possible to find a function $f(x)$, defined for all $1 \leq x < \infty$, such that $f(n) = a_n$: indeed, the function $f(x)$ whose graph is obtained by drawing straight line segments between (n, a_n), for $n = 1, 2, 3, \ldots$, is such a function.

Draw graph

Using this observation and Remark 1.2 we can use the limit laws of real-valued functions to immediately obtain the following helpful result.

Proposition 2.1 (Limit Laws for Sequences). *Let (a_n), (b_n) be convergent sequences, c a constant. Then,*

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$,
2. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$,
3. $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$,
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, *whenever* $\lim_{n\to\infty} b_n \neq 0$,
5. $\lim_{n\to\infty} a_n^r = (\lim_{n\to\infty} a_n)^r$, *if* $r > 0$ *and* $a_n > 0$, *for* $n = 1, 2, 3, \ldots$.

Create your own Theorem!

1. Use the Example from Section 1 and the Limit Laws to complete the following statement:

   Let $r >$ _____. Then, $\lim_{n\to\infty} \frac{1}{n^r} =$ _____.

2. Explain carefully why your completed statement is true.

Mathematical workout - flex those muscles

1. Consider the sequence (a_n), where $a_n = \frac{2n+1}{n+3}$.

   (a) Show that $a_n = \frac{2+\frac{1}{n}}{1+\frac{3}{n}}$.

   (b) Use the Limit Laws to give a careful explanation why $\lim_{n\to\infty} a_n$ exists and is equal to 2.

2. Explain carefully why the sequence (a_n) is convergent, where $a_n = \frac{3n^4+10n^2-1}{5n^{10}-2n^7+3n^2+n-1}$, and determine its limit.

Remark 2.2. In Problem Set 1 you will formulate and prove a statement determining precise conditions describing when a sequence (a_n) is convergent, where

$$a_n = \frac{\alpha_r n^r + \alpha_{r-1} n^{r-1} + \ldots + \alpha_1 n + \alpha_0}{\beta_s n^s + \beta_{s-1} n^{s-1} + \ldots \beta_1 n + \beta_0}$$

is a well-defined rational expression function of n. You will also determine conditions that allow you to find the limit of convergent sequences of this form.

3 The Squeeze Theorem

Definition 3.1. Let n be a natural number. We define the natural number $n!$, said n *factorial*, to be the product

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

By definition, we set $0! \stackrel{def}{=} 1$.

Example 3.2. We have $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

Consider the sequence (a_n), where

$$a_n = \frac{n!}{n^n}, \quad n = 1, 2, 3, \ldots.$$

Check your understanding

1. Write down the first five terms of the sequence (a_n).

2. Do you think the sequence is increasing, decreasing, bounded?

3. Do you think the sequence is convergent/divergent? If convergent, what do you expect the limit to be? If divergent, can you explain why?

Draw graph

Spot the pattern Consider the sequence (c_n), where $c_n = \frac{1}{n}$, and plot its graph above (along with the graph of (a_n)).

1. What is the relation between the first five terms of (a_n) and (c_n)?

2. What do you expect to be the relation between the n^{th} terms a_n and c_n?

3. Using the sequence (c_n), carefully explain, either, why (a_n) is convergent and what its limit is, or, why (a_n) is divergent.