Middlebury College

Calculus II: Fall 2017

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Sequences, an introduction II

1 Sequences, the basics

Definition 1.1. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. The collection of all outputs of the function $f(n)$ is called a **sequence**.

A sequence can be considered as an infinitely long list:

1	2	3	4	$\cdots$	n	$\cdots$
$f(1)$	$f(2)$	$f(3)$	$f(4)$	$\cdots$	$f(n)$	$\cdots$

We will frequently denote a sequence

$$(a_n)_{n\geq 1} = (a_1, a_2, a_3, a_4, \ldots, a_n, \ldots)$$

where $a_n = f(n)$. In particular, we *care about how we order the outputs of* $f(n)$.

Check your understanding

Consider the following real-valued functions

$$f(n) = \frac{p_n}{5}, \quad \text{where } p_n \text{ is the } n^{th} \text{ prime number},$$

$$g(n) = 2n + (-1)^n, \quad h(n) = \frac{1}{n}, \quad k(n) = (-1)^n.$$

1. Write down the first five terms of the corresponding sequence.
2. Plot the graph of the functions above.
3. What general features do the graphs exhibit? Describe as many as you can.
4. For each function above, give a property P of real numbers so that P holds for that function as $n \to \infty$.

Solution:

Remark 1.2. 1. Sequences will be denoted $(a_n)_{n\geq 1}$, or simply (a_n), where we assume *implicitly* that $a_n = f(n)$ for some real-valued function f whose domain is $\mathbb{N}$.

2. Given a sequence (a_n) such that $a_n = f(n)$, it is often useful to visualise the graph of $f(n)$ (in a similar manner as the above exercise). We will also call the graph of $f(n)$ **the graph of** (a_n).

3. Identifying a sequence (a_n) with a real-valued function $f(n)$ allows us to make sense of the following statement, where P is a property of real numbers:

$$\boxed{\text{property } P \text{ holds for } (a_n) \text{ as } n \to \infty.}$$

We now introduce some terminology for sequences.

Definition 1.3. Let (a_n) be a sequence.

1. The sequence (a_n) is **increasing** if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$. The sequence (a_n) is **strictly increasing** if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$

2. The sequence (a_n) is **decreasing** if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. The sequence (a_n) is **strictly decreasing** if $a_1 > a_2 > a_3 \cdots > a_n > \cdots$.

3. The sequence (a_n) is **bounded** if there exist real numbers m, M such that $m \leq a_n \leq M$, for every n. A sequence (a_n) is **unbounded** if it is not bounded.

Example 1.4. Let $a_n = g(n)$, where $g(n) = 2n + (-1)^n$. Then,

$$(a_n) = (1, 5, 5, 9, 9, 13, 13, \ldots)$$

Let n be a natural number. Then,

$$\begin{aligned} a_{n+1} - a_n &= g(n+1) - g(n) \\ &= 2(n+1) + (-1)^{n+1} - (2n + (-1)^n) \\ &= 2 + (-1)^{n+1} - (-1)^n \\ &= 2 + (-1)^n((-1) - 1) \\ &= 2 - 2(-1)^n = \begin{cases} 0, & \text{when } n \text{ even}, \\ 4, & \text{when } n \text{ odd}. \end{cases} \end{aligned}$$

In particular, for each natural number n, $a_{n+1} - a_n \geq 0$. Hence, (a_n) is increasing. However, (a_n) is *not* strictly increasing: for example, $a_2 \nless a_3$.

The sequence (a_n) is unbounded: it's impossible to find a real number M such that $a_n \leq M$, for every n. Indeed, observe that the even terms of (a_n) are precisely those positive integers having remainder 1 upon division by 4 - $5, 9, 13, 17, 21, \ldots$.

*A rigorous argument can be given as follows: suppose that such a real number M did exist, so that $a_n \leq M$, for all n. Let E be a multiple of 4 such that $E > M$. Then, $E = 4r$, for some integer r. Then,

$$a_{2r} = g(2r) = 2(2r) + (-1)^{2r} = 4r + 1 = E + 1 > E > M,$$

which violates the assumption that $a_{2r} \leq M$.

Check your understanding

1. For the functions $f(n), h(n), k(n)$ given above, determine which of the above attributes - (strictly) increasing/decreasing, (un)bounded - the corresponding sequences possess.

2. Let (a_n) be an increasing/decreasing sequence. Draw two different possible shapes of the graph of (a_n).

3. Let (a_n) be a bounded sequence. Draw two different possible shapes of the graph of (a_n).

Solution:

2 Limits Consider the increasing and bounded sequence (a_n), where $a_n = 10 - \frac{10}{n}$. On the graph of (a_n), consider the line $y = 10$.

True/False

1. If P is the property $P : 9 < y < 11$ then P holds for (a_n) as $n \to \infty$.

2. If P is the property P : *'the distance from y to* 10 *is less than* 0.01*'* then P holds for (a_n) as $n \to \infty$.

3. If P is the property $P : 9.9999 < y < 10.0001$ then P holds for (a_n) as $n \to \infty$.

4. If P is the property $P : 10 - \frac{1}{2^{100}} < y < 10 + \frac{1}{2^{100}}$ then P holds for (a_n) as $n \to \infty$.

5. Let $\varepsilon > 0$ be any positive real number. If P is the property

$$P : \text{'the distance from } y \text{ to } 10 \text{ is less than } \varepsilon\text{'}$$

then P holds for (a_n) as $n \to \infty$.

We will now formalise the situation observed above.

Let $\varepsilon > 0$ be a positive real number and let L be an arbitrary real number. We define $D_{\varepsilon,L}$ (note that this property depends upon both ε and L) to be the property of real numbers y,

$$D_{\varepsilon,L} : \text{'the distance from } y \text{ to } L \text{ is less than } \varepsilon\text{'}$$

Definition 2.1. Let (a_n) be a sequence. We say that (a_n) is a **convergent sequence with limit** L if, for any $\epsilon > 0$, the property $P_{\epsilon,L}$ holds for (a_n) as $n \to \infty$.

Equivalently, (a_n) is a **convergent sequence with limit** L if, for any $\epsilon > 0$, there is some natural number N such that

$$n \geq N \quad \Longrightarrow \quad |a_n - L| < \varepsilon.$$