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September 14 Lecture

Sequences, an introduction

1 The Natural Numbers. Define the natural numbers to be the collection of all positive integers

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \ldots$$

Denote by $\mathbb{N}$ the collection of natural numbers. The variable n will frequently be used to denote an arbitrary natural number.

In this lecture we will begin an investigation in to the behaviour of real-valued functions $f(n)$, having domain $\mathbb{N}$: this means that, to every natural number n we are associating a real number $f(n)$.

2 The phrase "n tends to ∞". Let $f(n)$ be some real-valued function, where n is a natural number. Here are some examples:

Example 2.1. ,

1. $f(n) = p_n$, where p_n is the n^{th} prime number; $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, etc.
2. $f(n) = n^2 + 2$.
3. $f(n) = (-1)^n$; $f(1) = -1$, $f(2) = 1$, $f(3) = -1$, etc.
4. $f(n) = 1 - \frac{1}{n}$; $f(1) = 0$, $f(2) = \frac{1}{2}$, $f(3) = \frac{2}{3}$, $f(4) = \frac{3}{4}$, etc.
5. $f(n) = n^{th}$ decimal digit of π; $f(1) = 1$, $f(2) = 4$, $f(3) = 1$, $f(4) = 5$, etc.
6. $f(n) = S_n$, where S_n is the n^{th} Riemann sum of the function $g(x) = x + 1$, $1 \leq x \leq 2$ (see September 11 Worksheet).
7. $f(n) =$ area of $K(n)$, where $K(n)$ is the n^{th} Koch snowflake (see September 11 Worksheet).

Suppose that P is some given property that we wish to check of a real number y: for example, P could be the property

$$`y > 1\text{'},$$

or the property

$$`\textit{the distance from } y \textit{ to } -\tfrac{1}{2} \textit{ is less than } 0.001\text{'}.$$

Consider the collection of all real numbers $y = f(n)$, where n is a natural number (i.e. the collection of *all outputs* of the function $f(n)$). Then, exactly one of the following three Conditions must hold:

(I) property P is true for $y = f(n)$, for *all but finitely many* n.

(II) property P is false for $y = f(n)$, *except for finitely many* n.

(III) neither (I) nor (II).

We will say that $f(n)$ **satisfies Condition (I), (II) or (III) for property** P.

CHECK YOUR UNDERSTANDING

Let P be the property '*y is an integer*'. Determine the Condition ((I), (II) or (III)) that the following functions satisfy for property P:

1. $f(n) = \frac{p_n}{5}$, where p_n is the n^{th} prime number.
2. $f(n) = \cos(n\pi)$.
3. $f(n) = 24\left(\frac{1+(-1)^n}{n}\right)$

Hint: it will be useful to write down $f(n)$ for some values of n; try to spot patterns!

Solution:

GET CREATIVE!

1. Provide an example of a property P and a function $f(n)$ so that $f(n)$ satisfies condition (I) for property P
2. Provide an example of a property P and a function $f(n)$ so that $f(n)$ satisfies condition (II)
3. Provide an example of a property P and a function $f(n)$ so that $f(n)$ satisfies condition (III)
4. Provide an example of two properties P and Q and a single function $f(n)$ so that $f(n)$ satisfies condition (I) for property P but it satisfies condition (II) for property Q.

Hint: make life simple for yourself; don't try to come up with overly elaborate properties or functions.

Solution:

We will now consider Condition (I) in more detail.

Let P be a property and $f(n)$ a function satisfying Condition (I) for property P. Then, there can exist at most finitely many (possibly zero!) inputs $n_1, \ldots, n_k$ such that property P **does not** hold for $f(n_1), \ldots, f(n_k)$. *Note:* the inputs $n_1, \ldots, n_k$ are not necessarily the first k natural numbers.

Let N be a natural number that is larger than each of $n_1, \ldots, n_k$. Then, for every $n \geq N$, property P holds for $f(n)$.

Example 2.2. Let P be the property '*y is* <u>*not*</u> *an integer*', and let $f(n) = \frac{p_n}{5}$, where p_n is the n^{th} prime number. Then, we have

$$f(1) = \frac{2}{5}, \quad f(2) = \frac{3}{5}, \quad f(3) = \frac{5}{5} = 1, \quad f(4) = \frac{7}{5}, \quad f(5) = \frac{11}{5}, \quad etc.$$

We see that property P holds for property P only if $n = 3$. Hence, if we take $N = 4$ then, for every $n \geq N$, property P holds for $f(n)$. In words, whenever n is greater than or equal to 4, the value $f(n) = \frac{p_n}{5}$ is not an integer.

We record our observation below.

To say that a function $f(n)$ satisfies Condition (I) for property P is precisely the same as saying that there exists some large natural number N so that, for every $n \geq N$, property P holds for $f(n)$.

CHECK YOUR UNDERSTANDING

Suppose that $f(n)$ satisfies Condition (I) for property P. Which of the following phrases are accurate:

1. property P holds for $f(n)$ for arbitrary n.
2. property P holds for $f(n)$ for sufficiently large n.
3. property P holds for $f(n)$ for $n \geq 1{,}000{,}000{,}000{,}000{,}000{,}000$.
4. property P eventually holds for $f(n)$.

Definition 2.3. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. Let P be a property. We say that **property P holds for $f(n)$ as n goes to infinity** if $f(n)$ satisfies Condition (I) for property P. We will often write **property P holds for $f(n)$ as $n \to \infty$**.

As a rigorous mathematical statement we have the following:

property P holds for $f(n)$ as $n \to \infty$ if there exists a natural number N such that, for every $n \geq N$, property P holds for $f(n)$.

3 Sequences

Definition 3.1. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. The collection of all outputs of the function $f(n)$ is called a **sequence**.

A sequence can be considered as an infinitely long list:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ f(1) & f(2) & f(3) & f(4) & \cdots & f(n) & \cdots \end{array}$$

We will frequently denote a sequence

$$(a_n) = (a_1, a_2, a_3, a_4, \ldots, a_n, \ldots)$$

where $a_n = f(n)$. In particular, we *care about how we* <u>*order*</u> *the outputs of* $f(n)$.

CHECK YOUR UNDERSTANDING

1. For the first four functions given in Example 2.1 write down the corresponding sequence.
2. Plot a graph of the first four functions given in Example 2.1.
3. What general features do the graphs exhibit? Describe as many features as you can.