Calculus II: Fall 2017

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Course Review Problems

The following resources provide additional problems:

- *Single Variable Calculus*, Stewart
- *Khan Academy*, https://www.khanacademy.org/math/calculus-home

$*$ = standard, similar to exam problems. $**$ = more difficult than exam problems.

Problems on Sequences & Series

Determine whether the sequence (a_n) or series $\sum b_n$ converges. Wherever possible, determine the limit.

1. $*$

$$a_n = \frac{2^n}{n^n}$$

2. $*$

$$\sum_{n=1}^{\infty} \frac{4^{n+1}}{10^{n-1}}$$

3. $*$

$$a_n = (-1)^n \frac{n}{n^3+1}$$

4. $*$

$$a_n = 2^n \cos(\pi n)$$

5. $*$

$$\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{\sqrt{n^3+1}}$$

6. $*$

$$\sum_{n=1}^{\infty} \frac{n}{4n-1}$$

7. $*$

$$\sum_{n=1}^{\infty} \frac{n^5}{5^n}$$

8. $*$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^3+1}$$

9. $*$

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

1. $*$ Define the sequence (a_n) by

$$a_1 = 1, \quad a_{n+1} = \sqrt{a_n + 6}, \quad \text{for } n \geq 1.$$

   (a) Using induction, show that (a_n) is increasing.

   (b) Using induction, show that $1 \leq a_n < 3$, for $n = 1, 2, 3, \ldots$.

   (c) Deduce that (a_n) is convergent and determine $L = \lim_{n\to\infty} a_n$.

2. $*$ Define the sequence (a_n) by

$$a_1 = 7, \quad a_{n+1} = \frac{a_n + 1}{4}, \quad \text{for } n \geq 1.$$

   (a) Using induction, show that (a_n) is decreasing.

   (b) Using induction, show that $0 < a_n \leq 7$, for $n = 1, 2, 3, \ldots$.

(c) Deduce that (a_n) is convergent and determine $L = \lim_{n\to\infty} a_n$.

3. $*$ Define the sequence (a_n) by

$$a_1 = 5, \quad a_{n+1} = \sqrt{2a_n + 3}, \quad \text{for } n \geq 1.$$

(a) Using induction, show that (a_n) is decreasing.

(b) Using induction, show that $3 \leq a_n \leq 5$, for $n = 1, 2, 3, \ldots$.

(c) Deduce that (a_n) is convergent and determine $L = \lim_{n\to\infty} a_n$.

Techniques of integration

Solve the following antiderivative problems.

1. $*$

$$\int \frac{x}{(x-1)^3}dx$$

2. $*$

$$\int x^3 \cos(x^2)dx$$

3. $**$

$$\int \frac{\sqrt{1-x^2}}{x^4}dx$$

4. $*$

$$\int \frac{x^3}{x^2-1}dx$$

5. $*$

$$\int \frac{\sin^3(x)}{\cos^7(x)}dx$$

6. $*$

$$\int \cos(3\log(x))dx$$

7. $*$

$$\int \frac{2x+3}{(x+1)^2}dx$$

8. $*$

$$\int \sqrt{5+4x-x^2}dx$$

9. $*$

$$\int \frac{x^2}{(5x^3-2)^{2/3}}dx$$

10. $*$

$$\int x \arcsin(x/2)dx$$

11. $*$

$$\int \frac{x}{(x^2+1)(x^2+2)}dx$$

12. $*$

$$\int \frac{1}{x^3\sqrt{x^2-1}}dx$$

Improper integrals

Determine whether the integral is convergent or divergent. Evaluate those that are convergent.

1. $*$

$$\int_0^\infty \frac{1}{x^2+x}dx$$

2. $*$

$$\int_0^\infty \sin^2(s)ds$$

3. $*$

$$\int_6^8 \frac{4}{(x-6)^3}dx$$

4. $*$

$$\int_0^\infty \frac{x\arctan(x)}{(1+x^2)^2}dx$$

5. $*$

$$\int_0^\infty \frac{1}{(2x+1)^3}dx$$

6. $*$

$$\int_0^\infty \frac{\arctan(x)}{2+\exp(x)}dx$$

7.$*$

$$\int_{-2}^{4} \frac{1}{x^4} dx$$

8.$*$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

9.$*$

$$\int_0^5 \frac{t}{t-2} dt$$

10.$*$

$$\int_0^\infty \frac{x}{x^3+1} dx$$

11.$*$

$$\int_{-\infty}^{\infty} (x^3 - 3x^2) dx$$

12.$*$

$$\int_0^1 \frac{\sec^2(x)}{x\sqrt{x}} dx$$

Differential Equations

Determine the general solution to the differential equation. If a given initial value is provided, determine the unique solution satisfying the initial condition.

1.$*$

$$xy' - y = x\log(x)$$

2.$*$

$$f'(x) = f(x)(1 - f(x))$$
$$f(0) = \frac{1}{2}$$

3.$*$

$$\frac{dx}{dt} = 1 - t + x - tx$$

4.$*$

$$\frac{dr}{dt} + 2tr = r$$
$$r(0) = 5$$

5.$*$

$$y' = \frac{xy\sin(x)}{y+1}$$

6.$*$

$$y' = x\exp(-\sin(x)) - y\cos(x)$$

7.$*$ (TYPO)

$$(1+\cos(x))y' = (1+\exp(-y))\sin(x)$$

8.$*$

$$y' = 3x^2\exp(y)$$
$$y(0) = 1$$

9.$*$

$$xy' = y$$

Power Series

Determine the centre and the interval of convergence for the power series.

1.$*$

$$\sum_{n=1}^{\infty} \frac{(-x)^n}{n^2 5^n}$$

2.$*$

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+2)!}$$

3.$*$

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n4^n}$$

4.$*$

$$\sum_{n=0}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$$

5.$*$

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{(2n+1)3^n}$$

6.$*$

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n}$$

7. ∗

$$\sum_{n=1}^{\infty} n^n x^n$$

8. ∗

$$\sum_{n=1}^{\infty} (-1)^n n 4^n x^n$$

9. ∗

$$\sum_{n=1}^{\infty} \frac{(3x+2)^n}{n^2}$$

Taylor Series

Determine the Taylor series of $f(x)$ at c, either using the definition of the Taylor series or by playing the '*Calculus Game*' and using known series representations of functions.

1. ∗

$$f(x) = \frac{x^2}{1+x}, \quad c = 0$$

2. ∗

$$f(x) = \log(4-x), \quad c = 0$$

3. ∗

$$f(x) = \frac{1}{(1-3x)^3}, \quad c = 0$$

4. ∗

$$f(x) = \sin(x), \quad c = \pi$$

5. ∗

$$f(x) = \sin(x), \quad c = \pi/2$$

6. ∗

$$f(x) = x^5 - 1, \quad c = 1$$

7. ∗

$$f(x) = \frac{1+x}{x-1}, \quad c = 0$$

8. ∗

$$f(x) = \frac{1}{1-x}, \quad c = -5$$

9. ∗

$$f(x) = \frac{1}{x}, \quad c = -3$$

Using Taylor's Inequality determine the values of x for which the Taylor series at $c = 0$ converges to $f(x)$.

1. ∗

$$f(x) = \cos\left(\frac{x}{2}\right)$$

2. ∗∗

$$f(x) = \frac{1}{3-2x}$$

3. ∗

$$f(x) = \exp(x+2)$$

Misc.

Show that the series is convergent and use an appropriate power series to determine its limit.

1. ∗

$$\sum_{n=4}^{\infty} (-1)^n \frac{2^n}{n!}$$

2. ∗

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$$

3. ∗

$$\sum_{n=0}^{\infty} (-1)^k \frac{\pi^{2k+1}}{(2k+1)!}$$

4. ∗

$$\sum_{n=1}^{\infty} (-1)^n \frac{(\log(2))^2}{n!}$$

5. ∗

$$\sum_{n=0}^{\infty} (-1)^k \frac{\pi^{2k}}{(2k)!}$$

6. ∗

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$$