Middlebury College

Calculus II: Fall 2017
Problem Set 6

Contact: `gmelvin@middlebury.edu`

Keywords: improper integrals, growth & decay equations, separable equations.

Problems for submission

A1. (a) Consider the improper integral

$$\int_0^\infty \frac{1}{(1+x^2)(1+\arctan(x))}dx \tag{$*$}$$

(i) Is this improper integral Type I or Type II? Explain carefully what it means to compute this integral.

(ii) Using method by substitution, determine whether the improper integral $(*)$ converges or diverges.

(iii) Use the Improper Integral Comparison Test to determine the convergence/divergence of $(*)$.

(b) Consider the improper integral

$$\int_0^1 \frac{4x}{\sqrt{1-x^4}}dx \tag{$**$}$$

(i) Is this improper integral Type I or Type II? Explain carefully what it means to compute this integral.

(ii) Using method by substitution, determine whether the improper integral $(**)$ converges or diverges.

A2. Consider the improper integral

$$\int_1^\infty \frac{1}{x\log(x)}dx$$

(a) Explain why this improper integral is both Type I and Type II.

(b) Explain, using limits, why we can split up the integral

$$\int_1^\infty \frac{1}{x\log(x)}dx = \int_1^2 \frac{1}{x\log(x)}dx + \int_2^\infty \frac{1}{x\log(x)}dx$$

(c) Determine whether

$$\int_1^\infty \frac{1}{x\log(x)}dx$$

converges or diverges.

A3. For the following improper integrals determine

(a) if they are Type I or Type II, and

(b) whether they converge or diverge.

(i)

$$\int_{-\infty}^\infty \frac{1}{\exp(x)+\exp(-x)}dx$$

(ii)

$$\int_1^\infty \frac{3x-1}{4x^3-x^2}dx$$

(*Hint: use partial fractions*)

(iii)

$$\int_1^\infty \frac{1+\sin(x)}{x^2}dx$$

(iv)

$$\int_1^2 \frac{1}{\sqrt{x^2-1}}dx$$

A4. A radioactive substance consists of atoms that undergo an instantaneous change. Every so often, an atom will emit a particle and change to another form. We call this a process of *radioactive decay*.

Define $y(t)$ to be the mass (measured in grams, say) of a radioactive substance at time t. For any one atom, it is impossible to predict when radioactive decay would occur. However, if we have very many atoms, on average some fraction, k, will undergo this decay during any instantaneous change in time (this fraction will depend on the material.) We call k the *decay constant*. This means that ky of the amount will be lost per unit time.

(a) Explain why $0 \leq k \leq 1$.

(b) Model the radioactive decay of a radioactive substance using a suitable Growth & Decay Equation.

(c) Suppose that initially there is y_0 grams of a radioactive substance. Determine the mass (in grams) of the radioactive substance at time t.

(d) The *half life* of a radioactive substance is the time $t = t_{HL}$ so that

$$y(t_{HL}) = \frac{y_0}{2}$$

Show that $t_{HL} = -\frac{\log(2)}{k}$.

(e) In 1986 the Chernobyl nuclear power plant exploded and scattered radioactive material over Europe. In particular, the two radioactive elements iodine-131 (whose half-life is 8 days) and cesium-137 (whose half life is 30 years) were emitted into the atmosphere.

i. What are the decay constants k_i and k_c associated to iodine-131 and cesium-137 respectively?

ii. Determine the number of days it takes for iodine-131 to decay to 0.01% of its initial level.

iii. Determine the number of years it takes for cesium-137 to decay to 0.01% of its initial level.

A5. The *(normalised) logistic equation* is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

This differential equation describes *restricted population growth* of a population $P(t)$: when $P(t)$ is small (relative to M) there is rapid population growth and as $P(t)$ approaches M the rate of growth slows. Here $k > 0$ is a constant of proportionality measuring the relative instantaneous growth of the population and M is the called the *carrying capacity*.

For example, the logistic equation provides a more accurate model of the spread of a disease: M is the total number of people (there is a limit to the number of people that can get the disease).

(a) Using separation of variables show that the general solution to the logistic equation is

$$P(t) = \frac{M}{1 + C\exp(-kt)}$$

(b) Describe what happens to the population $P(t)$ as $t \to \infty$.

(c) Let $P(0) = P_0$ be the initial size of the population. Show that $C = \frac{M - P_0}{P_0}$.

(d) What happens as $t \to \infty$ to the population $P(t)$ if $P_0 < M$? What happens as $t \to \infty$ to the population $P(t)$ if $P_0 > M$? What happens as $t \to \infty$ to the population $P(t)$ if $P_0 = M$? Draw a graph showing the general behaviour of $P(t)$ in each of these cases.

Additional recommended problems (not for submission)

B1. Determine the convergence of following improper integrals. If possible, determine the limit. A problem marked with !!! indicates you should be careful with your approach.

a) $\int_0^2 \frac{2x}{4-x^2}dx$ b) !!! $\int_0^2 \frac{1}{\sqrt{(x-1)^3}}dx$ c) $\int_1^\infty \frac{\sin^2(x)}{x^2}dx$ d) $\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{\cos(x)}}dx$

e) $\int_{-1}^2 \frac{1}{\sqrt{x+1}}dx$ f) $\int_{-2}^2 \frac{1}{x^4}dx$ g) $\int_1^\infty \frac{x}{(x+1)^3}dx$ h) $\int \frac{3}{x^2+5}dx$

B2. Determine c such that

$$x^3\exp(-2x) \le \exp(-x), \quad \text{for all } x \ge c$$

Hence, show that the improper integral

$$\int_0^\infty x^3\exp(-2x)dx$$

converges using the Improper Integral Comparison Test.

Challenging Problems

C1. (*)