Middlebury College

Calculus II: Fall 2017
Problem Set 2

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Problems for submission

A1. For the following problem you should *not* make use of any online resource (e.g. Wikipedia).

(a) Recall the construction of the Koch snowflake $K(m)$ (see September 11 Worksheet, Problem 2, available online) from an equilateral triangle having side length $1m$. Given that the number of edges of $K(n) = 3 \cdot 4^n$ (*you do need to show this*), show that the area of the m^{th} Koch snowflake $K(m)$ is

$$\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\sum_{i=1}^{m}\left(\frac{4}{9}\right)^i$$

(*You may want to use the September 11 Worksheet solutions; in particular, Problem 2f*)

(b) Define $K(\infty)$ to be the *infinite Koch snowflake* - this is the planar region obtained as the limit as $m \to \infty$ of the Koch snowflakes $K(m)$. In particular,

$$\text{area of } K(\infty) = \lim_{m\to\infty} (\text{area of } K(m))$$

Show that

$$\text{area of } K(\infty) = \frac{2\sqrt{3}}{5}$$

(c) Show that the perimeter of $K(m)$ has length $3 \cdot \left(\frac{4}{3}\right)^m$. Explain why the infinite Koch snowflake $K(\infty)$ has perimeter of '*infinite length*'(!).

A2. Determine whether the series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{n+5}{2n^3+1}$ b) $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ c) $\sum_{i=1}^{\infty} \frac{6^i}{5^i-1}$ d) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ e) $\sum_{n=1}^{\infty} \frac{1}{2n+3}$

f) $\sum_{n=1}^{\infty} \frac{1}{3^n+4^n}$ g) $\sum_{m=1}^{\infty} \frac{1-(-1)^m}{m^4}$ h) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ i) $\sum_{k=1}^{\infty} \frac{1}{2^k k!}$ j) $\sum_{j=1}^{\infty} \frac{1+j!}{(1+j)!}$.

A3. In this problem you will investigate the relationship betwen repeating decimals and fractions.

(a) Let $x = 0.9999\ldots$. You may have seen the claim that $x = 1$.

i. Express x as a geometric series.
ii. Is this geometric series convergent or divergent? If convergent, determine its limit; if divergent, give a careful justification.
iii. How many decimal representation does the number 1 have?
iv. Which numbers have more than one decimal representation?

(b) Express the following numbers as a ratio of integers:

i. $0.88888\ldots$
ii. $0.12121212\ldots$
iii. $0.3464646\ldots$
iv. $2.53165165165\ldots$

(c) i. Make use of the Geometric Series Theorem to show that

$$0.abababab\ldots = \frac{10a+b}{99}$$

ii. Determine digits a, b such that

$$0.ababab\ldots = \frac{24}{99}$$

A4. For each of the following alternating series determine whether the series is absolutely convergent, conditionally convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ c) $\sum_{n=4}^{\infty} \frac{n!}{(-100)^n}$ d) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ e) $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}}$ f) $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$

A5. Consider the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$.

(a) Explain carefully why the series is convergent.

(b) Determine a partial sum s_m that approximates the limit of the series to within 4 decimal places.

Additional recommended problems (not for submission)

B1. In this problem you will determine formulae for general geometric series. Consider the Geometric Series Theorem and its preceding discussion (September 21 Lecture). Let $|r| < 1$ and a be a constant.

(a) Determine a formula for the series $\sum_{n=0}^{\infty} ar^n$.

(b) Determine a formula for the series $\sum_{n=5}^{\infty} ar^n$. (*Hint: show that the partial sums t_m of this series are related to the partial sums s_m of the series $\sum_{n=0}^{\infty} ar^n$ by the relation $t_m = r^5 s_m$.*)

(c) Let $k \geq 0$ be an integer. Determine a formula for the series $\sum_{n=k}^{\infty} ar^n$.

(d) What happens if $|r| > 1$? What happens if $r = \pm 1$?

B2. Find the values of x for which the series converges. Find the sum of the series for those values of x.
a) $\sum_{n=1}^{\infty} (-5)^n x^n$ b) $\sum_{n=1}^{\infty} (x+1)^n$ c) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$ d) $\sum_{n=1}^{\infty} \frac{2^n}{x^n}$

B3. The *Fibonacci sequence* $(f_n)_{n\geq 1}$ is the sequence defined recursively by

$$f_1 = 1, \quad f_2 = 1, \qquad f_n = f_{n-1} + f_{n-2}, \quad n = 3, 4, 5, \ldots$$

Show that each of the following statements is true.

(a) $\frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n-2}}$

(b) $\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 1$

(c) $\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n-2}} = 2$

B4. Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where

$$b_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ odd,} \\ \frac{1}{n^2}, & \text{if } n \text{ even,} \end{cases}$$

is divergent. Why does the Alternating Series Test not apply?

B5. Let r be a real number.

(a) Show that the series $\sum_{n=0}^{\infty} \frac{r^2}{(1+r^2)^n}$ is always convergent and determine the limit (*Hint: geometric series*). Explain why you must consider the cases $r = 0$ and $r \neq 0$ separately.

(b) For which r is the series $\sum_{n=0}^{\infty} \frac{r}{(1+r)^n}$ convergent? Determine the limit.

Challenging Problems

C1. In this problem you will investigate the *Cantor set* $\mathcal{C}$. We will construct the Cantor set $\mathcal{C}$ by a recursive process. The Cantor set is named after Georg Cantor (1845-1918), one of the most important figures in the history of mathematics.

Denote by $[a, b]$ the *closed line segment* having endpoints $x = a$ and $x = b$: $[a, b]$ is the collection of real numbers x satisfying $a \leq x \leq b$. Denote by (a, b) the *open line segment* bounded by $x = a$ and $x = b$: (a, b) is the collection of real numbers x satisfying $a < x < b$.

Let $C(0)$ denote the closed unit interval $[0, 1]$, the collection of real numbers x satisfying $0 \leq x \leq 1$. We think of $C(0)$ as a line segment having length 1.

Define $C(1)$ to be the collection of real numbers obtained from $C(0)$ by removing the open line segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, so that $C(1)$ is the collection of real numbers x satisfying $0 \leq x \leq \frac{1}{3}$ or $\frac{2}{3} \leq x \leq 1$. We call this procedure '*removal of the middle thirds*'. Hence, we can think of $C(1)$ as consisting of the two closed line segments $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. We say that the *total length of* $C(1)$ is the sum of the lengths of its closed line segments: hence, the total length of $C(1)$ is $\frac{2}{3}$

Having constructed $C(k)$, $k = 1, 2, 3, \ldots$, we construct $C(k+1)$ by '*removing the middle thirds*' from each of the line segments that comprise $C(k+1)$. We say that the *total length of* $C(k)$ is the sum of the lengths of its closed line segments.

(a) Write down the line segments that comprise $C(2)$. Show that the total length of $C(2)$ is $\frac{4}{9}$.

(b) Let $k > 2$. Write down the line segments that comprise $C(k)$. Show that the total length of $C(k)$ is $\frac{2^k}{3^k}$.

(c) Define the Cantor set $\mathcal{C}$ to be the collection of real numbers $0 \leq x \leq 1$ for which x is a member of $C(k)$, for every $k = 1, 2, 3, \ldots$. Show that $\mathcal{C}$ does not contain any open line segment (a, b).

(d) Describe an infinite subcollection of $\mathcal{C}$. Deduce that $\mathcal{C}$ is an infinite collection

(e) A *base*-3 expansion of a real number x is a representation of x as the sum of a series

$$x = \sum_{n=1}^{\infty} \frac{c_n}{3^n},$$

where c_n is only allowed to equal $0, 1, 2$. We will also denote a base-3 expansion of x by

$$x = [0.c_1c_2c_3c_4\ldots]_3.$$

For example,

$$\frac{1}{4} = 0 \cdot 3^{-1} + 2 \cdot 3^{-2} + 0 \cdot 3^{-3} + 2 \cdot 3^{4} + \ldots = [0.020202\ldots]_3$$

This can be checked by multiplying both sides by 4: indeed,

$$8 \cdot 3^{-2} + 8 \cdot 3^{-4} + \ldots = 8 \sum_{n=1}^{\infty} \frac{1}{3^2} = 8 \left(\frac{1}{9} \cdot \frac{1}{1 - \frac{1}{9}} \right) = 1.$$

Show that $\mathcal{C}$ consists of precisely those real numbers $0 \leq x \leq 1$ that admit a base-3 expansion consisting of only 0's and 2's.

C2.