Calculus II: Fall 2017

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Practice Examination III

Trigonometric Identities, Integral Formulae and Power Series
You may use the following identities and formulae without proof.

a.

$$\cos^2(x) + \sin^2(x) = 1$$

$$1 + \tan^2(x) = \sec^2(x)$$

b.

$$\begin{aligned} \sin(2x) &= 2\sin(x)\cos(x) \\ \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 1 - 2\sin^2(x) \\ &= 2\cos^2(x) - 1 \end{aligned}$$

c.

$$\int \sec(x)dx = \log(\sec(x) + \tan(x)) + C$$

$$\int \tan(x)dx = \log|\sec(x)| + C$$

d.

$$\tfrac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\tfrac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

e.

$$\int \tfrac{1}{\sqrt{a^2-x^2}}dx = \tfrac{1}{a}\arcsin\left(\tfrac{x}{a}\right) + C$$

$$\int \tfrac{1}{a^2+x^2}dx = \tfrac{1}{a}\arctan\left(\tfrac{x}{a}\right) + C$$

f.

$$\begin{aligned} \sin(x) &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \dots \\ &= \textstyle\sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k+1)!}x^{2k+1} \end{aligned}$$

g.

$$\begin{aligned} \cos(x) &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \dots \\ &= \textstyle\sum_{k=0}^{\infty} \tfrac{(-1)^k}{(2k)!}x^{2k} \end{aligned}$$

h.

$$\begin{aligned} \log(1+x) &= x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \dots \\ &= \textstyle\sum_{k=0}^{\infty} \tfrac{(-1)^k}{k}x^k \end{aligned}$$

Instructions:

- Write your name on this exam and any extra sheets you hand in.
- Sign the Honor Code Pledge below.
- You will have 180 minutes to complete this Examination.
- You **must** attempt Problem 1.
- You **must** attempt **at least five** of Problems 2, 3, 4, 5, 6, 7, 8.
- If you attempt more than five problems from Problems 2, 3, 4, 5, 6, 7, 8, then your final score will be the sum of your score for Problem 1 and the highest possible score obtained from five of the remaining problems.

- There are 5 blank pages attached for scratchwork.
- **Calculators are not permitted.**
- **Explain your answers *clearly* and *neatly* and in *complete English sentences*.**
- State all Theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.

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1. (20 points) True/False:

   (a) $\sum_{n=2}^{\infty} \frac{4^n}{7^{n+1}}$ is convergent.

   (b) Let $A = (-1, 2]$. The function $f(x) = \frac{1-x}{1+x}$ with domain A admits an inverse function.

   (c) Let $e = \exp(1)$ be Euler's number. Then,

   $$\int_1^{\exp(3)} \frac{dt}{t} = e^3$$

   (d) The function $y = \frac{1}{1+x}$ is a solution to the differential equation

   $$y' = y + xy.$$

   (e) The function $f(x) = \frac{5}{3+x}$ admits the power series representation

   $$\frac{5}{3} - \frac{5}{3}x + \frac{5}{3}x^2 - \frac{5}{3}x^3 + \dots \quad \text{whenever } |x| < 1.$$

   (f) Let $f(x) = \frac{1}{1-x}$ and let $R_n(x)$ be the n^{th}-degree remainder of its Taylor series centred at $c = 0$. Then,

   $$|R_n(x)| \leq \frac{1}{n+1}, \qquad \text{whenever } |x| \leq \frac{1}{2}.$$

   (g) The series $\sum_{n=1}^{\infty} \frac{\sin(n)-1}{n^3+2}$ is convergent.

   (h)

   $$\int_0^2 \frac{1}{(1-x)^2} dx = -2$$

   (i) Let $f(x) = \frac{1}{2-x}$. Any power series representation of $f(x)$ has radius of convergence $R = 2$.

   (j) Let $f(x)$ be a solution of the differential equation

   $$f'(x) = \frac{2x}{f(x)}$$

   The set of points $(x, f(x))$ in the plane describes an ellipse.

2. Solve the following integration problems. Show all your working and provide complete justification.

   (a) (10 points)

   $$\int \frac{1}{\sqrt{x^2 + 4x + 3}} dx$$

(b) (10 points)

$$\int 2x \log(x+1)dx$$

3. (a) (10 points) Let $x \neq 1$ be a real number. Consider the mathematical proposition

$$P(n): \quad 1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2} \qquad n \geq 1$$

Using induction, show $P(n)$, for all $n \geq 1$.

(b) (10 points) By considering the sequence of partial sums of the power series $\sum_{n=1}^{\infty} nx^{n-1}$, use problem (a) to determine all x for which the series $\sum_{n=1}^{\infty} nx^{n-1}$ converges.

4. Consider the function $f(x)$ with Taylor series centred at $c = 3$

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{12^n n}$$

(a) (10 points) Determine the interval of convergence of the power series.

(b) (5 points) What is $f'''(3)$?

(c) (5 points) Determine the Taylor series centred at $c = 3$ of $f'(x)$, the derivative of $f(x)$.

5. (a) (10 points) Determine whether the following improper integral is convergent or divergent. If convergent, determine its limit.

$$\int_0^2 \frac{1}{4-x^2}dx$$

(b) (10 points) Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(3n)!}$$

6. (a) (10 points) Determine the solution to the initial value problem

$$xy' + y = x^2 + 1, \quad y(1) = 1$$

(b) (10 points) Determine the general solution to the differential equation

$$y' = x - 4xy$$

7. Consider the sequence $(a_n)_{n\geq 1}$ where

$$a_1 = \sqrt{7}, \qquad a_{n+1} = \sqrt{a_n + 7}, \quad n = 1, 2, 3, \ldots$$

(a) (5 points) Using induction, show that (a_n) is increasing.

(b) (5 points) Using induction, show that $a_n \leq 4$, for $n = 1, 2, 3, \ldots$.

(c) (5 points) Explain carefully why (a_n) is a convergent sequence.

(d) (5 points) Determine $L = \lim_{n\to\infty} a_n$.

8. (a) (10 points) Determine the Taylor series of $f(x) = 3\sin(x-1)$ centred at $c = 1$.

(b) (10 points) Determine those x for which the Taylor series you obtained in part (a) equals $f(x)$.