Middlebury College

Calculus II: Fall 2017

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October 9 Lecture

An exp-traordinary function II

Today we continue our investigation of the exp-traordinary function. We will investigate the differentiability of $\exp(x)$ and show that it satisfies a particular *differential equation*. Next lecture we will see how this differential equation is related to the problem of finding the *inverse function of* $\exp(x)$.

1 The exponential function Recall the function

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Last lecture you determined the following properties of $\exp(x)$:

- $\exp(x) > 1$, for any real number $x > 0$.
- $\exp(0) = 1$.
- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) > 0$, for any real number x.
- $\exp(x + y) = \exp(x) \cdot \exp(y)$, for any real numbers x, y. $(*)$

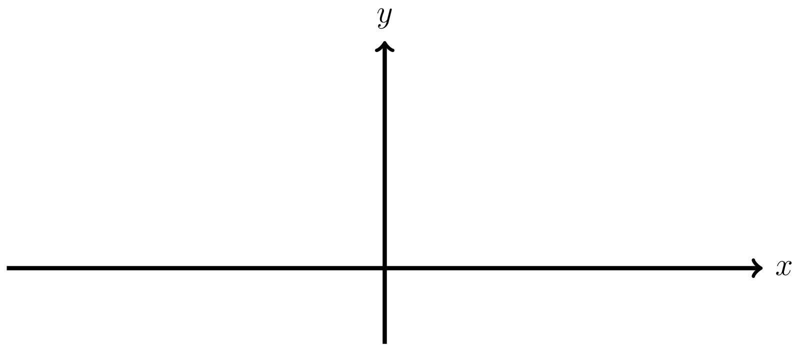
We call property $(*)$ the **Remarkable Property of** $\exp(x)$.

Moreover, the function $\exp(x)$ is **strictly increasing**:
if $x < y$, so that $y = x + h$ with $h > 0$, then

$$\begin{aligned} \exp(y) &= \exp(x + h) \\ &= \exp(x)\exp(h), \qquad \text{by } (*), \\ &> \exp(x), \qquad \text{since } \exp(h) > 1. \end{aligned}$$

Check your understanding

Based on these investigations, draw the graph of the function $\exp(x)$.



Remark 1.1.

1. Determining the value $\exp(x)$, for a given real number $x \neq 0$, is **difficult**: this requires our being able to determine the limit of the series

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

2. Observe that

$$\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$$

This series is a series with positive terms, which implies that its sequence of partial sums (s_m) is strictly increasing. In particular, for any $m = 0, 1, 2, \dots$,

$$s_m < \exp(1) \quad \text{and} \quad \lim_{m\to\infty} s_m = \exp(1).$$

Notice that $s_2 = 1 + 1 + \frac{1}{2} = \frac{5}{2}$ and

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{1}{2} . \frac{1}{1 - \frac{1}{2}} = 1$$

Hence,

$$2.5 = \frac{5}{2} = s_2 < \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots < 1 + 1 + 1 = 3$$

so that

$$2.5 < \exp(1) < 3.$$

3. In Problem Set 4 you will have the opportunity to show that

$$\exp(1) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

and, more generally,

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

Define **Euler's number** to be the limit

$$e \overset{def}{=} \exp(1).$$

Then, in fact, it can be shown that

$$\exp(x) = e^x.$$

2 O Calculus, Where Art Thou? Let $-1 < h < 1$ and consider the series

$$\frac{\exp(h) - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{h^n}{(n+1)!}.$$

CHECK YOUR UNDERSTANDING

1. Let $a_n = \frac{h^n}{(n+1)!}$. Use the ratio test to show that the series $1+\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.

2. As h gets close to 0, describe what happens to the expression

$$\frac{\exp(h) - 1}{h}$$

3. Complete the following statement

$$\lim_{h \to 0} \frac{\exp(h) - 1}{h} = \underline{\hspace{3cm}}$$

Recall what it means for a function $f(x)$ to be *differentiable at* $x = a$: we say that $f(x)$ is **differentiable at** $x = a$ if the following limit exists

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

In this case we write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If $f(x)$ is differentiable for every input value x, then we define the **derivative of** $f(x)$ to be the function

$$f'(x) \stackrel{def}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We also write

$$\frac{d}{dx} f(x) = f'(x).$$

4. Let a be a real number. Using the Remarkable Property, show that

$$\frac{\exp(a+h) - \exp(a)}{h} = \exp(a) \left(\frac{\exp(h) - 1}{h} \right)$$

5. Use the above formula to deduce that

$$\exp'(a) = \exp(a), \qquad \text{for every real number } a.$$

6. Complete the following statement:

Let a be a real number. Then, $\exp(x)$ is ____________________ at __________

Moreover,

$$\frac{d}{dx} \exp(x) = \_\_\_\_\_\_\_\_\_\_$$