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October 6 Lecture

An exp-traordinary function

In today's lecture we will define a very interesting function. Investigating this function will lead us to the notion of an *inverse function*, and will take our mathematical journey back to the familiar calculus realm of differentiation and integration.

1 Defining a function via a series Let x be any real number and consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Mathematical workout - flex those muscles

Use the ratio test to show that the above series is (absolutely) convergent, for every real number x.

By assigning to every real number x the limit of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have provided the definition for a *well-defined* function

$$(\text{INPUT}) \qquad x \quad \mapsto \quad \exp(x) \overset{def}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (\text{OUTPUT})$$

We will call the function $\exp(x)$ just defined **the exponential function**. Let's investigate some of its basic properties.

Check your understanding

Using the definition of $\exp(x)$, show that $\exp(0) = 1$, $\exp(x) > 1$, for any $x > 0$, and $\exp(x) > 1 + x$, for any $x > 0$.

2 A remarkable property We are going to investigate a remarkable property of the exponential function. Let x be a real number. For each $m = 0, 1, 2, \ldots$, denote the m^{th} partial sum of the series $\exp(x)$ by $s_m(x)$, so

$$s_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

Check your understanding

Let x be a real number.

1. Write down the expressions for $s_0(x)$, $s_1(x)$, $s_2(x)$, $s_3(x)$.

2. Show that $s_1(x)s_1(y) = s_1(x+y) +$ *higher order terms.*

3. Show that $s_2(x)s_2(y) = s_2(x+y) +$ *higher order terms.*

4. **Guess the pattern!** Complete the following statement

$$s_3(x)s_3(y) = \underline{\hspace{4cm}} + \textit{ higher order terms}$$

5. **Guess the general pattern!** Complete the following statement: for every $k = 0, 1, 2, \ldots$

$$s_k(x)s_k(y) = \underline{\hspace{4cm}} + \textit{ higher order terms}$$

Recall that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty} s_n(x).$$

Complete the following statement:

$$\boxed{\exp(x) \cdot \exp(y) = \underline{\hspace{5cm}} \qquad (*)}$$

Property $(*)$ has lots of remarkable consequences. For example, suppose that x is any positive real number. Then,

$$\begin{aligned} 1 &= \exp(0) \\ &= \exp(x + (-x)) \\ &= \exp(x) \cdot \exp(-x) \end{aligned}$$

In particular,

- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) \neq 0$, for any real number x.

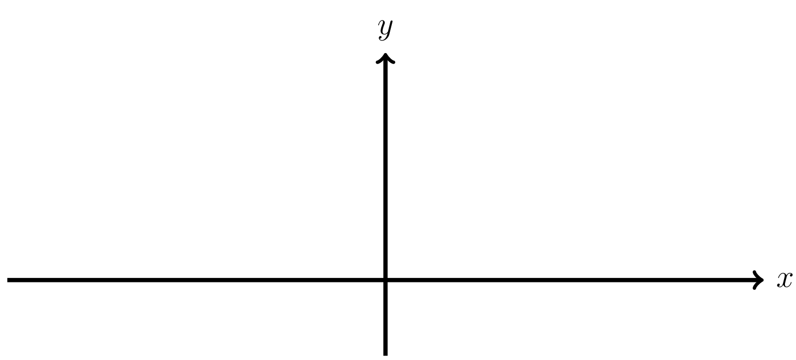
Check your understanding

1. Let x be a real number and write $x = 2y = y + y$. Use $(*)$ to show that $\exp(x) = \exp(2y) \geq 0$. Deduce that $\exp(x) > 0$, for every real number x. (*Hint: use* $\exp(2y) = \exp(y + y)$.)

2. Let $x < y$ and write $y = x + h$, where $h > 0$. Use $(*)$ to show that $\exp(y) > \exp(x)$. (*Hint: recall that* $\exp(h) > 1$ *whenever* $h > 0$)

   Hence, the exponential function is **strictly increasing**.

3. Based on your investigations, draw the graph of the function $\exp(x)$.



Summary

- $\exp(x+y) = \exp(x) \cdot \exp(y)$, for any real numbers x, y.
- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) > 0$, for any real number x.
- $\exp(x)$ **is a strictly increasing function**. $(**)$

We will discuss the consequences of Property $(**)$ in the next Lecture. In particular, over the next week or so we will show the following

- $\exp(x)$ has a *functional inverse*: there is a function $L(y)$ satisfying

$$\exp(L(y)) = y, \qquad L(\exp(x)) = x.$$

- $\exp(x)$ is a differentiable function; hence, $\exp(x)$ is a continuous function.
- $\exp(x)$ is the solution of a *differential equation*:

$$\frac{d}{dx} f(x) = f(x).$$

Remark 2.1. 1. Observe that

$$\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$$

This series is a series with positive terms, which implies that its sequence of partial sums (s_m) is strictly increasing. In particular, for any $m = 0, 1, 2, \ldots$,

$$s_m < \exp(1) \quad \text{and} \quad \lim_{m \to \infty} s_m = \exp(1).$$

Notice that $s_2 = 1 + 1 + \frac{1}{2} = \frac{5}{2}$ and

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{2} . \frac{1}{1 - \frac{1}{2}} = 1$$

Hence,

$$2.5 = \frac{5}{2} = s_3 < \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < 1 + 1 + 1 = 3$$

so that

$$2.5 < \exp(1) < 3.$$

2. In Problem Set 4 you will have the opportunity to show that

$$\exp(1) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

and, more generally,

$$\exp(x) = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$