Calculus II: Fall 2017

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October 4 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart: Section 11.5.
- *AP Calculus BC*, Khan Academy: Estimating infinite series.

Approximating real numbers

In today's lecture we will discuss approaches to determining approximations of real numbers using sequences and series.

1 Approximations and the ratio test Recall the following series from the September 28 Lecture

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad (*)$$

Let $b_n = \frac{n^2}{2^n}$. On September 28, we determined the following facts:

Fact 1. For $n \geq 4$, $b_n \leq b_3 \left(\frac{8}{9}\right)^{n-3}$.

Fact 2. The series $\sum_{n=1}^{\infty} b_n$ is convergent with limit L.

Basic question: what is L?
Better question: can we approximate L?

Definition 1.1. Let $\sum a_n$ be a *convergent* series having positive terms with limit L. Let (s_m) be the associated sequence of partial sums. Define the **remainder after** m **terms** to be

$$R_m = L - s_m = \sum_{n=m+1}^{\infty} a_n.$$

Check your understanding

1. Let $\sum a_n$ be a series and suppose that $R_{10} < 0.001$. Explain why s_{10} is an approximation to L, correct to 2 decimal places.

2. For the series $(*)$ above, express R_3 and R_5 as a series.

3. Use Fact 1 to show that

$$R_3 \le b_3 \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n \qquad R_5 \le b_3 \sum_{n=3}^{\infty} \left(\frac{8}{9}\right)^n \qquad R_{10} \le b_3 \sum_{n=8}^{\infty} \left(\frac{8}{9}\right)^n$$

4. Using the formula (Problem Set 2, B1)

$$\sum_{n=k}^{\infty} r^k = \frac{r^k}{1-r}, \qquad \text{whenever } -1 < r < 1,$$

show that $b_3 \sum_{n=k}^{\infty} \left(\frac{8}{9}\right)^k = \frac{8^{k-1}}{9^{k-2}}$.

5. How closely do the partial sums s_3 and s_5 approximate L? How closely does s_{10} approximate L? (*You may want to use the calculator on your mobile device*)

STOP! Await further instructions.

The investigation above can be generalised to obtain the following
Ratio Test Approximation Theorem (RTAT)

> Let $\sum a_n$ be a series of positive terms and let $r_n = \frac{a_{n+1}}{a_n}$. Suppose that $l = \lim_{n\to\infty} r_n < 1$, so that $\sum a_n$ converges by the Ratio Test.
>
> - If (r_n) is a decreasing sequence and $r_{n+1} < 1$ then
>
> $$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}.$$
>
> - If (r_n) is an increasing sequence then
>
> $$R_n \le \frac{a_{n+1}}{1 - l}$$

Example 1.2. Consider the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$. Letting $a_n = \frac{3^n}{n!}$, we have

$$r_n = \frac{a_{n+1}}{a_n} = \frac{3^{n+1} \cdot n!}{(n+1)! \cdot 3^n} = \frac{3}{n+1} \to 0 < 1 \quad \text{as } n \to \infty.$$

Hence, the series is convergent by the Ratio Test. The sequence (r_n) is decreasing and $r_n < 1$ whenever $n \ge 3$. Hence, by RTAT, the remainder after n terms R_n satisfies the estimate

$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}} = \frac{3^{n+1}}{(n+1)! \cdot (1 - \frac{3}{n+2})} = \frac{3^{n+1}(n+2)}{(n+1)! \cdot (n-1)}$$

Therefore, to approximate the limit L of the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ to within 3 decimal places it suffices to determine n so that

$$\frac{3^{n+1}(n+2)}{(n+1)! \cdot (n-1)} < 0.0001,$$

as then s_n will be correct to 3 decimal places of L. For example, if $n = 13$ then we have

$$\frac{3^{14} \cdot 15}{14! \cdot 12} = 0.000068580275751034679606\ldots < 0.0001,$$

and $s_{13} = 19.0854\ldots$ is an approximation to L correct to 3 decimal places.

Mathematical workout - flex those muscles!

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

1. Let $a_n = \frac{1}{n2^n}$. Show that

$$r_n = \frac{a_{n+1}}{a_n} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right)$$

2. Explain why the series $\sum a_n$ is convergent.

3. Explain why the sequence (r_n) is increasing. Show that $\lim_{n\to\infty} r_n = \frac{1}{2}$.

4. Use RTAT to show that the remainder after n terms R_n has the estimate

$$R_n \le \frac{1}{(n+1)2^n}.$$

5. Show that the 10^{th} partial sum s_{10} provides an approximation of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ correct to 3 decimal places.

6. Determine the limit $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ to 3 decimal places.