



OCTOBER 2 LECTURE

SUPPLEMENTARY REFERENCES:

- *Discrete Mathematics & its Applications*, Rosen: Section 3.2.
- *Series & Induction*, Khan Academy: Induction.

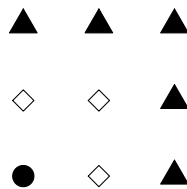
MATHEMATICAL INDUCTION

GET CREATIVE!

We are going to investigate a formula for the sum \mathcal{O}_n of the first n odd natural numbers.

$$\mathcal{O}_n = 1 + 3 + 5 + 7 + 9 + \dots + (2n - 1)$$

1. Determine $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$.
2. Guess the value of \mathcal{O}_n .
3. Use the following diagram to explain the value you obtained for \mathcal{O}_3 .



4. Using the above diagram, give a similar visual representation of \mathcal{O}_4 .
5. Using your guess for \mathcal{O}_n , explain how you could obtain the value for \mathcal{O}_{n+1} . (*You can either explain in words and/or make us of a diagram to justify your explanation*)

1 Mathematical induction

Many mathematical theorems are of the form

$$P(n), \text{ for every natural number } n. \quad (*)$$

Here $P(n)$ is some mathematical proposition that depends on n .

Example 1.1. Here are some examples of mathematical propositions $P(n)$ dependent on n .

1. $P(n)$: $\sum_{i=0}^n 2^i = 2^{n+1} - 1$.
2. $P(n)$: $n < 2^n$.
3. $P(n)$: for every real number $r \neq 1$, $r + r^2 + r^3 + \dots + r^n = \frac{r(1-r^n)}{1-r}$.

CHECK YOUR UNDERSTANDING

Complete the following mathematical proposition:

$P(n)$: the sum of the first _____ natural numbers is _____

BRAIN TEASER!

Consider the following situation: Suppose that we have an infinite line of people, called Person 1, Person 2, Person 3, ... etc. We have also been provided with the following information.

- B. It is known that Person 1 has received a Snapchat message.
- I. It is also known that, whenever Person k receives a Snapchat message they will immediately send a Snapchat message to Person $k + 1$.

CHECK YOUR UNDERSTANDING

Provided with the information B and I, we will investigate the following question:

Q: is it possible that there is some person who does not receive a Snapchat?

1. What is your preliminary (i.e. after < 5 seconds thought) answer to the question above: YES or NO?
2. Let's suppose the answer is YES. This means there is some person, Person k say, who does not receive a Snapchat. Assume, further, that Person k is the least such person to not receive a Snapchat i.e. if Person l does not receive a Snapchat then $l \geq k$. Explain why Person $k - 1$ must have received a Snapchat message.
3. Explain why, given all of the information above, Person k must have both received *and* not received a Snapchat.
4. What do you now believe is the answer to *Q*?

The proof technique known as **mathematical induction** provides a valid approach to proving statements of the form (*). It proceeds as follows:

1. **BASE CASE.** Show directly that the proposition $P(1)$ is true.
2. **INDUCTIVE STEP.** Show that the statement

$$\text{'if } P(n) \text{ then } P(n+1)\text{'}$$

holds for any natural number n .

Determining both the **BASE CASE** and the **INDUCTIVE STEP** is *sufficient* to obtain $P(n)$, for every natural number n . The reasoning is similar to the Snapchat message problem above.

Remark 1.2. The assumption ‘if $P(n)$ ’ in the **INDUCTIVE STEP** is called the **inductive hypothesis**.

Example 1.3. Let $P(n)$ be the proposition

$$\sum_{i=0}^n 2^i = 2^{n+1} - 1.$$

We will show, by mathematical induction, that $P(n)$ holds, for every natural number n .

BASE CASE: $P(1)$ is the proposition

$$P(1): \sum_{i=0}^1 2^i = 2^2 - 1.$$

This holds since the left hand side is $1 + 2 = 3 = 2^2 - 1$. hence, the **BASE CASE** is true.

INDUCTIVE STEP: Suppose that $P(k)$ is true, for a natural number k . We want to show that this implies that $P(k+1)$ is true: this will verify the **INDUCTIVE STEP**.

So, assume that the proposition $P(k)$ is true, for a natural number k . Hence, we know that

$$\sum_{i=0}^k 2^i = 2^{k+1} - 1.$$

Consider the proposition

$$P(k+1): \sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1.$$

We want to show that this proposition holds true. Now, the left hand side of the above expression is

$$\sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^k 2^i.$$

Since we know, by assumption, that $P(k)$ is true, we can rewrite the right hand side of this expression to get

$$\begin{aligned} \sum_{i=0}^{k+1} 2^i &= 2^{k+1} + (2^{k+1} - 1) \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1. \end{aligned}$$

Hence, if $P(k)$ holds then $P(k + 1)$ holds.

Therefore, by mathematical induction, $\sum_{i=0}^n 2^i = 2^{n+1} - 1$, for every natural number n .

CHECK YOUR UNDERSTANDING

Prove the following statements using mathematical induction.

1. The sum of the first n odd integers is n^2 , for every natural number n .
2. For every natural number n , $n < 2^n$. Use your result to deduce that the alternating series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$ is convergent.