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October 27 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.4.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Techniques of Integration VII. Partial Fractions contd.

In this lecture we complete our discussion of the method of partial fractions. We will investigate how to proceed when the denominator contains irreducible quadratic factors.

We recall the notion of an irreducible quadratic polynomial and the fact that every polynomial with real coefficients factorises as a product of linear polynomials and irreducible quadratic polynomials. This (surprisingly difficult!) algebraic fact will allow us to *solve the antiderivative problem for any rational function*.

1 Don't stop completin' (the square) In this paragraph we will recall that oft-forgotten method of completing the square. This method allows us to take an arbitrary quadratic (i.e. degree 2) polynomial $ax^2 + bx + c$ and write it in the form

$$A(x + B)^2 + C$$

for appropriate constants A, B, C. Let's look at some examples.

Mathematical workout - flex those muscles!

1. Rewrite $x^2 + 2x$ in the form $A(x + B)^2 + C$, for appropriate A, B, C.

2. Rewrite $x^2 + 2x + 2$ in the form $A(x + B)^2 + C$, for appropriate A, B, C.

3. Rewrite $x^2 + x$ in the form $A(x + B)^2 + C$, for appropriate A, B, C.

4. Rewrite $x^2 + x + 2$ in the form $A(x + B)^2 + C$, for appropriate A, B, C.

Completing the square

$$ax^2 + bx + c = A\left(x + B\right)^2 + C$$

where

$$A = a, \qquad B = \frac{b}{2a}, \qquad C = c - \frac{b^2}{4a}$$

2 Irreducible quadratic polynomials: an algebraic interlude

Let $f(x)$ be a quadratic polynomial,

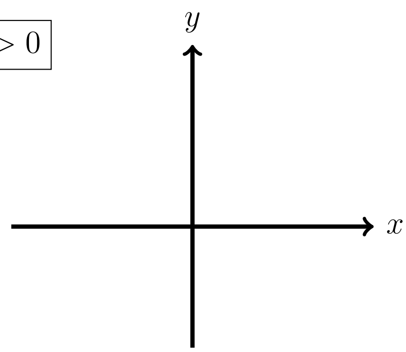
$$f(x) = ax^2 + bx + c = A(x + B)^2 + C$$

If A and C both have the same sign (in particular, $C \neq 0$) then $f(x)$ does not admit real roots.

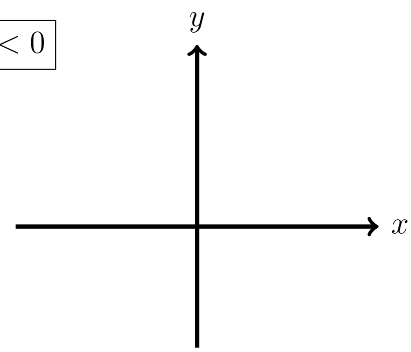
CHECK YOUR UNDERSTANDING

Draw the graph of $f(x)$, making sure you mark down its vertex and line of symmetry.

$\boxed{A, C > 0}$



$\boxed{A, C < 0}$



In this case, we say that $f(x)$ is an **irreducible quadratic** polynomial function. By the Remainder Theorem, this means that the *quadratic polynomial* $f(x)$ **does not factorise as a product of linear factors**.

Remainder Theorem

Let $p(x)$ be a polynomial. If $p(a) = 0$ then $p(x) = (x - a)q(x)$.

Example 2.1. By completing the square, you have shown above that

$$f(x) = x^2 + x + 2 = \underline{\hspace{4cm}}$$

Hence, since $A =$ _____, $C =$ _____, $f(x)$ is an irreducible quadratic.

CHECK YOUR UNDERSTANDING

Complete the following statements: Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial function.

- If $a > 0$ and $b^2 - 4ac$__________ then $f(x)$ is an irreducible quadratic.
- If $a < 0$ and $b^2 - 4ac$__________ then $f(x)$ is an irreducible quadratic.

We will make use of the following theorem in algebra:

Polynomial factorisation theorem:

Let $f(x)$ be a polynomial function. Then, $f(x)$ can be factorised as a product of linear polynomials and irreducible quadratic polynomials.

3 Partial fractions: the case of irreducible quadratic factors Now that we have reminded ourselves of the notion of an irreducible quadratic polynomial, let's get back to calculus!

> **Aim:** Solve the antiderivative problem
> $$\int f(x)dx$$
> where $f(x) = \frac{P(x)}{Q(x)}$ is a rational function, and $Q(x)$ contains irreducible quadratic factors.

Example 3.1. Determine

$$\int \frac{2x^2+1}{x^3+2x^2+2x}dx$$

As the degree of the numerator of the integrand is less than the degree of the denominator, we do not need to perform long division. We factorise the denominator

$$x^3+2x^2+2x = \underline{\hspace{3cm}}$$

We try for a partial fraction decomposition of the form

$$\frac{2x^2+1}{x^3+2x^2+2x} = \frac{A}{\underline{\hspace{1cm}}} + \frac{Bx+C}{\underline{\hspace{2cm}}}$$

Taking the common denominator of the right hand side and equating numerators, we obtain

$$2x^2+1 = A(\underline{\hspace{3cm}}) + (Bx+C)(\underline{\hspace{2cm}})$$

Remember, this is an equality of *functions*, so we can input values for x on either side of this equation.

$$\text{Input } x=0\text{:} \quad 1 = \underline{\hspace{3cm}} \implies A = \underline{\hspace{2cm}}$$
$$\text{Input } x = \underline{\hspace{1cm}}\text{:} \qquad \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$$
$$\text{Input } x = \underline{\hspace{1cm}}\text{:} \qquad \underline{\hspace{2cm}} = \underline{\hspace{3cm}}$$

Solving for B, C, we find

$$B = \underline{\hspace{2cm}}, \quad C = \underline{\hspace{2cm}}$$

Hence,

$$\int \frac{2x^2+1}{x^3+2x^2+2x}dx = \int \underline{\hspace{3cm}}dx + \int \underline{\hspace{4cm}}dx$$

Using the substitution $u = x+1$, so that $\frac{du}{dx} = 1$, we have

$$\underline{\hspace{3cm}} = \frac{3u-5}{u^2+1} \cdot \frac{du}{dx}$$

Hence, the method of substitution gives

$$
\begin{aligned}
\int \underline{\hspace{4cm}}\, dx &= 3\int \frac{u}{u^2+1}du - 5\int \frac{1}{u^2+1}du \\
&= \frac{3}{2}\ln(u^2+1) - 5\arctan(u), \quad \text{using sub. } v = u^2+1 \text{ for first integral,} \\
&= \frac{3}{2}\ln((x+1)^2+1) - 5\arctan(x+1) + C
\end{aligned}
$$

Combining everything we have

$$
\begin{aligned}
\int \frac{2x^2+1}{x^3+2x^2+2x} &= \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{2}\int \frac{3x-2}{(x+1)^2+1}dx \\
&= \frac{1}{2}\ln|x| + \frac{3}{4}\ln((x+1)^2+1) - \frac{5}{2}\arctan(x+1) + C
\end{aligned}
$$

Phew...!

Check your understanding

Complete the following steps to determine

$$\int \frac{2x^2-x+1}{x^3+x}dx$$

1. Determine constants A, B, C such that

$$\frac{2x^2-x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

2. Use the previous problem to determine

$$\int \frac{2x^2-x+1}{x^3+x}dx$$

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems **using the method of partial fractions**. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. $\int \frac{x^3+4}{x^2+4} dx$
2. $\int \frac{x^4+1}{x^5+4x^3} dx$
3. $\int \frac{x^2}{x^2+x+2} dx$
4. Challenge! What do you think the partial fraction decomposition of

$$\frac{1}{x(x^2+4)^2}$$

looks like? (*Hint: what did we do for repeated linear factors?*)