Calculus II: Fall 2017

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.4.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Techniques of Integration VI. Partial Fractions contd.

In this lecture we extend our discussion of the method of partial fractions. We will gain exposure to the different aspects of this method that can appear.

1 Partial fractions: the case of distinct linear factors The method of partial fractions we saw yesterday can be extended to the case when the denominator of a rational function can be factored as a product of *distinct linear factors*.

Example 1.1. Determine

$$\int \frac{2x^2+1}{(x^2-1)(x^2-4)}dx$$

As the degree of the numerator is less than the degree of the denominator we do not need to perform long division. Observe that

$$(x^2-1)(x^2-4) = \underline{\hspace{5cm}},$$

is a product of *distinct linear factors*. Based on our investigations yesterday, we might expect to be able to write

$$\frac{2x^2+1}{(x^2-1)(x^2-4)} = \frac{2x^2+1}{\underline{\hspace{5cm}}} = \frac{A}{\underline{\hspace{2cm}}} + \frac{B}{\underline{\hspace{2cm}}} + \frac{C}{\underline{\hspace{2cm}}} + \frac{D}{\underline{\hspace{2cm}}}$$

Rearranging and taking acommon denominator leads to the equality of numerators

$$2x^2+1 = A(x+1)(x^2-4) + B(x-1)(x^2-4) + C(x^2-1)(x+2) + D(x^2-1)(x-2)$$

This last equality is an *equality of functions*: inputting a value for x on either side will give the same output. In particular,

$$\begin{aligned}
\text{Input } x=1: &\quad 2.1^2+1 = \underline{\hspace{3cm}} \implies A = \underline{\hspace{2cm}}\\
\text{Input } x=-1: &\quad 2.(-1)^2+1 = \underline{\hspace{3cm}} \implies B = \underline{\hspace{2cm}}\\
\text{Input } x=2: &\quad 2.2^2+1 = \underline{\hspace{3cm}} \implies C = \underline{\hspace{2cm}}\\
\text{Input } x=-2: &\quad 2.(-2)^2+1 = \underline{\hspace{3cm}} \implies D = \underline{\hspace{2cm}}
\end{aligned}$$

Method of Partial Fractions Case I: distinct linear factors

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function, where

$$Q(x) = (a_1x + a_2)(b_1x + b_2)\cdots(p_1x + p_2)$$

is a product of *distinct* linear factors (i.e. none of the factors repeat and no factor is a constant multiple of another).

- If $\deg P(x) \geq \deg Q(x)$ perform long division and write

$$\frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Proceed to the next step, replacing $P(x)$ by $r(x)$.

- If $\deg P(x) < \deg Q(x)$ determine constants $A, B, \dots, P$ so that

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x + a_2} + \frac{B}{b_1x + b_2} + \dots + \frac{P}{p_1x + p_2}$$

It is a fact that the constants $A, B, \dots, P$ you determine are unique.

Check your understanding

Let $a \neq 0$. Determine

$$\int \frac{1}{x^2 - a^2}dx$$

Remark 1.2. In general, it may not be so easy to determine if $Q(x)$ can be written as a product of distinct linear factors. For example, the polynomial

$$2x^3 - x^2 - 13x - 6$$

is a product of distinct linear factors: can you see how to factorise it?

More generally, it is a difficult problem to figure out how to factorise an arbitrary polynomial.

2 Partial fractions: repeated linear factors In this paragraph we will extend the method of partial fractions to the case of *repeated linear factors*. The problem here is that if we try to mimic the approach above, there are several possible routes to take.

Mathematical workout - flex those muscles!

Consider the rational function

$$f(x) = \frac{2x-1}{(x-1)^2}$$

1. Determine a constant A so that

$$f(x) = \frac{A}{x-1} + \frac{x}{(x-1)^2}$$

2. Determine constants B, C so that

$$f(x) = \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

3. Which of the above expressions for $f(x)$ do you think is easier to integrate? Back up your choice and try to determine

$$\int f(x)dx$$

Example 2.1. Determine

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$$

Since the degree of the numerator is larger than the degree of the denominator, we must perform long division. We find

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = \underline{\hspace{6em}} + \frac{}{x^3 - x^2 - x + 1}$$

Let's factor the denominator $Q(x) = x^3 - x^2 - x + 1$. Since $Q(1) = 0$, we must have that $x - 1$ is a factor of $Q(x)$ (this is the Remainder Theorem from high school algebra). Performing long division gives

$$\frac{x^3 - x^2 - x + 1}{x - 1} = x^2 - 1 \quad \Longrightarrow \quad x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)^2(x+1)$$

We try look to find constants A, B, C so that

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Taking the common denominator on the right hand side of the above equation gives

$$4x = A(x^2 - 1) + B(x + 1) + C(x - 1)^2$$

This is an equality of functions, so inputting a value for x on either side will give the same output. We find

$$\text{Input } x = 1: \quad 4.1 = \underline{\hspace{6em}} \Longrightarrow \underline{\hspace{2em}} = \underline{\hspace{4em}}$$

$$\text{Input } x = -1: \quad 4.(-1) = \underline{\hspace{6em}} \Longrightarrow \underline{\hspace{2em}} = \underline{\hspace{4em}}$$

How to determine _____?

Hence,

$$\frac{4x}{(x-1)^2(x+1)} = \frac{}{x-1} + \frac{}{(x-1)^2} + \frac{}{x+1}$$

and

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx = \underline{\hspace{20em}}$$

Method of Partial Fractions Case II: repeated linear factors

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function, where

$$Q(x) = (a_1x + a_2)^r(b_1x + b_2)^s \cdots (c_1x + c_2)^t$$

is a product of (possibly repeated) linear factors.

- If $\deg P(x) \geq \deg Q(x)$ perform long division and write

$$\frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Proceed to the next step, replacing $P(x)$ by $r(x)$.

- If $\deg P(x) < \deg Q(x)$ determine constants $A_1, \ldots, A_r, B_1, \ldots, B_s, \ldots, C_1, \ldots, C_t$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + a_2} + \frac{A_2}{(a_1x + a_2)^2} + \ldots + \frac{A_r}{(a_1x + a_2)^r} + \frac{B_1}{b_1x + b_2} + \ldots + \frac{B_s}{(b_1x + b_2)^s} + \ldots$$

It is a fact that the constants $A_1, \ldots, A_r, B_1, \ldots, B_s, \ldots, C_1, \ldots, C_t$ you determine are unique.

Check your understanding

Determine

$$\int \frac{1 - 2x^2}{x^4 + 2x^3 + x^2} dx$$

by completing the following steps.

1. Factorise $x^4 + 2x^3 + x^2$ as a product of linear factors.

2. Determine a partial fraction decomposition

$$\frac{1 - 2x^2}{x^4 + 2x^3 + x^2} = \underline{\hspace{6cm}}$$

3. Use the previous problems to obtain

$$\int \frac{1-2x^2}{x^4+2x^3+x^2}dx$$

3 Don't stop completin' (the square) In this paragraph we will recall that oft-forgotten method of completing the square. This method allows us to take an arbitrary quadratic (i.e. degree 2) polynomial ax^2+bx+c and write it in the form

$$A(x+B)^2+C$$

for appropriate constants A, B, C. Let's look at some examples.

Mathematical workout - flex those muscles!

1. Rewrite x^2+2x in the form $A(x+B)^2+C$, for appropriate A, B, C.

2. Rewrite x^2+3x+3 in the form $A(x+B)^2+C$, for appropriate A, B, C.

3. Rewrite x^2+x+5 in the form $A(x+B)^2+C$, for appropriate A, B, C.

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems **using the method of partial fractions**. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. $\int \frac{x}{x^4+x^3-2x^2} dx$

2. $\int \frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} dx$

3. $\int \frac{x^3-2x^2-4}{x^3-2x^2} dx$

4. Challenge! Consider $f(x) = \frac{2x-1}{(x-1)^2}$ from p.3. Rewrite

$$f(x) = \frac{2x-2}{(x-1)^2} + \frac{A}{(x-1)^2}$$

for a constant A. What method can you use to determine $\int f(x)dx$ in this case? Is this approach different to what we did in class?