Calculus II: Fall 2017

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.4.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Techniques of Integration V. Partial Fractions.

In this lecture we will focus on the method of partial fractions. This method allows us to split up a rational function as a sum of elementary rational functions. In this way we will devise an approach to solving the antiderivative problem for a large class of rational functions.

1 Rational functions

Definition 1.1. Let $f(x)$ be a function.

1. We say that $f(x)$ is a **polynomial function** if it can be written
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$
where $a_n, \ldots, a_0$ are constants. If $a_n \neq 0$ then we say that $f(x)$ **has degree** n and write $\deg f(x) = n$.

2. We say that $f(x)$ is a **rational function** if it can be written
$$f(x) = \frac{P(x)}{Q(x)},$$
where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x)$ is nonzero.

Example 1.2. The following functions are polynomials having degree $3, 5, 8$
$$f(x) = 2x^3 - x^2 + 10x + 1, \qquad g(x) = 5x^5 + 3x - 1, \qquad h(x) = 7x^8 + 2x^2.$$
The following functions are rational functions
$$\frac{7x^8 + 2x^2}{2x^3 - x^2 + 10x + 1}, \qquad \frac{5x^5 + 3x - 1}{7x^8 + 2x^2}, \qquad \frac{5x^5 + 3x - 1}{2x^3 - x^2 + 10x + 1}$$

Recall the method of **long division of polynomial functions**:

> Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function where $\deg P(x) \geq \deg Q(x)$. Then, there exists polynomial functions $b(x)$ and $r(x)$, where $\deg r(x) < \deg Q(x)$ so that
> $$f(x) = \frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Example 1.3. Consider the rational function

$$f(x) = \frac{4x^4 - 6x^3 + x - 3}{2x^2 - x + 4}$$

Then, the method of long division of polynomial proceeds as follows (it's analogous to long division of integers - replace x by 10):

$$\begin{array}{r|llllll}
 & & & 2x^2 & -2x & -5 \\ \hline
2x^2 - x + 4 & 4x^4 & -6x^3 & & +x & -3 \\
 & -4x^4 & +2x^3 & -8x^2 & & \\ \cline{2-4}
 & & -4x^3 & -8x^2 & +x & \\
 & & 4x^3 & -2x^2 & +8x & \\ \cline{3-5}
 & & & -10x^2 & +9x & -3 \\
 & & & 10x^2 & -5x & +20 \\ \cline{4-6}
 & & & & 4x & +17
\end{array}$$

Hence, $b(x) = 2x^2 - 2x - 5$ and $r(x) = 4x + 17$. You can check that

$$\frac{4x^4 - 6x^3 + x - 3}{2x^2 - x + 4} = 2x^2 - 2x - 5 + \frac{4x + 17}{2x^2 - x + 4}$$

Check your understanding

Perform long division on the following rational function

$$\frac{2x^3 + x^2 - 4}{x^2 - 1}$$

Complete the following statement

$$\boxed{\frac{2x^3 + x^2 - 4}{x^2 - 1} = \underline{\hspace{4cm}} + \frac{\hspace{4cm}}{x^2 - 1}}$$

2 Method of partial fractions

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function.

$$\boxed{\textbf{Goal: } \text{determine the (indefinite) integral} \quad \int f(x)dx}$$

Check your understanding

Complete the following steps to determine

$$\int \frac{2x^3 + x^2 - 4}{x^2 - 1} dx$$

You've shown above that

$$\boxed{\frac{2x^3 + x^2 - 4}{x^2 - 1} = 2x + 1 + \frac{2x - 3}{x^2 - 1}}$$

Hence, the difficulty lies in determining $\int \frac{2x-3}{x^2-1} dx$.

1. Find constants A and B so that

$$A(x + 1) + B(x - 1) = 2x - 3.$$

(*Hint: equate coefficients of powers of x*)

2. Observe that $x^2 - 1 = (x - 1)(x + 1)$. Use the previous problem to complete the following statement

$$\boxed{\frac{2x^3 + x^2 - 4}{x^2 - 1} = \underline{\hspace{6em}} + \frac{\hspace{4em}}{x - 1} + \frac{\hspace{4em}}{x + 1}}$$

3. Deduce

$$\boxed{\int \frac{2x^3 + x^2 - 4}{x^2 - 1} dx = \hspace{20em}}$$

The process of splitting up the rational function $f(x) = \frac{2x^3+x^2-4}{x^2-1}$ into a sum of simpler rational functions is known as the **method of partial fractions**.

Example 2.1. Determine

$$\int \frac{x^4 + 2x}{x^2 - 3x + 2} dx$$

First, we perform long division to obtain

$$
\begin{array}{rrrrrr}
 & & & x^2 & +3x & +7 \\
\cline{2-6}
x^2-3x+2 \big) & x^4 & & & +2x & \\
 & -x^4 & +3x^3 & -2x^2 & & \\
\cline{2-4}
 & & 3x^3 & -2x^2 & +2x & \\
 & & -3x^3 & +9x^2 & -6x & \\
\cline{3-5}
 & & & 7x^2 & -4x & \\
 & & & -7x^2 & +21x & -14 \\
\cline{4-6}
 & & & & 17x & -14
\end{array}
$$

Hence,

$$\frac{x^4+2x}{x^2-3x+2} = x^2+3x+17+\frac{17x-14}{x^2-3x+2}$$

and we need to determine

$$\int \frac{17x-14}{x^2-3x+2}dx$$

Observe that $x^2-3x+2=(x-2)(x-1)$. We want to find constants A and B such that

$$\frac{17x-14}{x^2-3x+2} = \frac{A}{x-2}+\frac{B}{x-1}$$

Multiplying both sides of this equation by x^2-3x+2 gives

$$17x-14 = A(x-1)+B(x-2) = (A+B)x+(-A-2B)$$

Hence, A, B satisfy the simultaneous equations

$$\begin{array}{rcl} A+B & = & 17 \\ -A-2B & = & -14 \end{array}$$

Adding the first equation to the second equation gives $B=-3$. Substituting $B=-3$ into the first equation gives $A=20$. Hence,

$$\frac{17x-14}{x^2-3x+2} = \frac{20}{x-2}-\frac{3}{x-1}$$

Therefore,

$$\begin{aligned} \int \frac{x^4+2x}{x^2-3x+2}dx &= \int \left(x^2+3x+17+\frac{20}{x-2}-\frac{3}{x-1}\right)dx \\ &= \frac{x^3}{3}+\frac{3}{2}x^2+17x+20\log(x-2)-3\log(x-1)+C \end{aligned}$$

In tomorrow's lecture we will formalise and generalise the approach we have taken today so that we can handle more complicated rational functions.

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems **using the method of partial fractions**. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. $\int \frac{x}{x^2+x-2}dx$
2. $\int \frac{x^6}{x^2-4}dx$
3. $\int \frac{x^2+1}{(x-1)(x+1)(x-2)}dx$