Middlebury College

Calculus II: Fall 2017

Contact: gmelvin@middlebury.edu

October 20 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.3.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Techniques of Integration IV. Inverse Trigonometric Substitutions.

Today we continue our investigation into inverse trigonometric substitutions. We will see lots of examples.

1 Inverse trigonometric substitution, again The method of inverse trigonometric substitution proceeds as follows: we are looking to determine

$$\int h(x)dx$$

where $h(x)$ contains one of the following expressions

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2} \qquad \text{where } a \text{ is some constant.}$$

Strategy:

1. Make the following substitution, depending on which of the above expressions appears in $h(x)$:

$x = a\sin(t)$	$\leftrightarrow$	$\sqrt{a^2 - x^2}$
$x = a\tan(t)$	$\leftrightarrow$	$\sqrt{a^2 + x^2}$
$x = a\sec(t)$	$\leftrightarrow$	$\sqrt{x^2 - a^2}$

2. Suppose we've made the substitution $x = T(t)$ above. Write

$$h(T(t))\frac{dx}{dt} = f(t)$$

3. Determine

$$\int f(t)dt$$

4. Substitute $t = T^{-1}(t)$ into the resulting expression.

Check your understanding

Let's determine

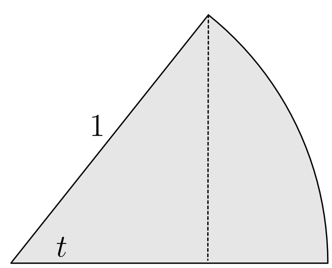
$$\int \frac{1}{x\sqrt{36-x^2}}dx$$

by the method of inverse trigonometric substitution.

1. Let $x = 6\sin(t)$. Show that

$$\frac{1}{x\sqrt{36-x^2}}\frac{dx}{dt} = \frac{1}{6}\csc(t)$$

2. Given that $x = 6\sin(t)$, complete the following triangle:



3. Recall that

$$\csc(t) = \frac{1}{\sin(t)}, \qquad \cot(t) = \frac{\cos(t)}{\sin(t)}.$$

Verify the following derivatives:

- $\frac{d}{dt}\cot(t) = -\csc^2(t)$
- $\frac{d}{dt}\csc(t) = -\csc(t)\cot(t)$
- $\frac{d}{dt}\log(\csc(t) - \cot(t)) = \csc(t)$

Deduce that

$$\int \csc(t)dt = \underline{\hspace{6cm}}$$

4. Use the above triangle to complete the following statements:

• $\csc(t) = \underline{\hspace{4cm}}$
• $\cot(t) = \underline{\hspace{7cm}}$

5. Combine your answers above to determine

$$\int \frac{1}{x\sqrt{36 - x^2}}dx$$

Hint: you have all the pieces of the puzzle, now see how the ***Strategy*** *tells you to fit them together.*

Check your understanding

Let's determine

$$\int \frac{1}{x^2\sqrt{x^2 + 1}}dx$$

by the method of inverse trigonometric substitution.

1. Let $x = \tan(t)$. Show that

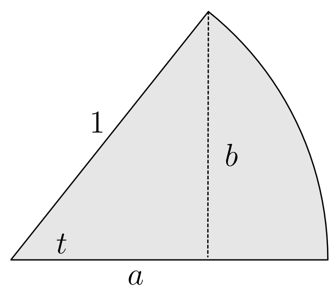
$$\frac{1}{x^2\sqrt{x^2 + 1}} \frac{dx}{dt} = \frac{\sin(t)}{\cos^2(t)}$$

2. Determine

$$\int \frac{\cos(t)}{\sin^2(t)} dt$$

Hint: use an appropriate substitution.

3. Given that $x = \tan(t)$, use Pythagoras' Theorem to determine a, b.



4. Combine your answers above to determine

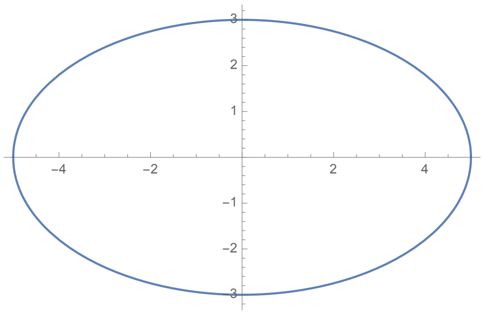
$$\int \frac{1}{x^2\sqrt{x^2+1}} dx$$

2 Total ellipse of the heart Let's apply what we've learned. An **ellipse** is a curve in the plane described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{$*$}$$

The following diagram is the ellipse determined by the equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$



Rearranging ($*$), we see that the portion of the ellipse lying in the upper half-plane (i.e. $y \geq 0$) is described by the equation

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Using symmetry, we find that the area A enclosed by the ellipse is

$$A = 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}dx$$

We are now able to determine this area. Let $x = a\sin(t)$. Then, we compute

$$\sqrt{a^2 - x^2}\frac{dx}{dt} = a^2\cos^2(t)$$

Hence, using the method of inverse trigonometric substitution,

$$4\frac{b}{a}\int \sqrt{a^2 - x^2}dx = 4ab\int \cos^2(t)dt = 4ab \cdot \frac{1}{2}(t + \frac{\sin(2t)}{2}) + C$$

In order to recover x from t we write $t = \arcsin(\frac{x}{a})$. We defined arcsin so that its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Hence, if $x = a$ then $t = \arcsin(1) = \frac{\pi}{2}$ and $x = 0$ then $t = \arcsin(0) = 0$. Therefore,

$$A = 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}dx = 4ab\int_{t=0}^{t=\pi/2} \cos^2(t)dt = 2ab\left[t + \frac{\sin(2t)}{2}\right]_{t=0}^{t=\pi/2} = \pi ab$$

Therefore, the area enclosed by the ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab. In particular, if $a = b$ then the ellipse is a circle and we recover the area of a circle having radius a.

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems **using the method of inverse trigonometric substitution**. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. $\int x^3\sqrt{1-x^2}dx$

2. $\int \frac{\sqrt{x^2-4}}{x}dx$. *Use the substitution* $x = 2\sec(t)$ *and recall that* $\sec^2(t) = \tan^2(t) + 1$.