Middlebury College

# Calculus II: Fall 2017

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# October 19 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.3.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

## Techniques of Integration III. Inverse Trigonometric Substitutions.

Today we will begin to investigate the method of *inverse trigonometric substitution*.

**1 Trigonometric integrals** You saw yesterday how to approach integrals of the form

$$\int \sin^n(x)\cos^m(x)dx$$

where $m, n \geq 0$ are integers, at least one of which is odd. What if both $m$ and $n$ are even? How can we determine the integral in this case?

Mathematical workout - flex those muscles!

Recall the trigonometric formulae

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$1 = \cos^2(x) + \sin^2(x)$$

Use these formulae to complete the following statements:

- $\sin^2(x) = \_\_\_\_\_ - \_\_\_\_\_ \cos(2x)$
- $\cos^2(x) = \_\_\_\_\_ + \_\_\_\_\_ \cos(2x)$

Check your understanding

Determine

$$\int \cos^2(x)dx$$

**Remark 1.1.** In Problem Set 5 you will investigate integrals of the form

$$\int \tan^n(x)\sec^m(x)dx$$

where $m, n \geq 0$ are integers. As we will see later in the course, these trigonometric integrals arise frequently in computations of *arc length*: for example, in determining the perimeter of an ellipse.

A useful trigonometric identity is

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \qquad (*)$$

CHECK YOUR UNDERSTANDING

1. Use $(*)$ and an identity above to show that

$$\cos^2(x)\sin^2(x) = \frac{1}{8}(1-\cos(4x))$$

2. Determine

$$\int \cos^2(x)\sin^2(x)dx$$

The identity $(*)$ is a special case of the following trigonometric identities

- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
- $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES!

1. Use the identities

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

to show that

- $\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$

2. Complete the following identities:

- $\sin(x)\cos(y) = \frac{1}{2}\left(\_\_\_\_\_\_\_\_\_\_\_\_ + \sin(x+y)\right)$
- $\sin(x)\sin(y) = \frac{1}{2}\left(\_\_\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_\_\_\right)$

**Example 1.2.** The above identities are very useful. For example, if we are looking to determine

$$\int \sin(5x)\cos(4x)dx$$

then we can write

$$\sin(5x)\cos(4x) = \frac{1}{2}\left(\sin(x) + \sin(9x)\right)$$

Hence,

$$\int \sin(5x)\cos(4x)dx = -\frac{1}{2}\left(\cos(x) + \frac{1}{9}\cos(9x)\right) + C$$

**2 Inverse trigonometric substitution** In this paragraph we will develop an approach to determine integrals whose integrands contain an expression of the form

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2}$$

This is known as the *method of inverse trigonometric substitution*.

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES

Using the identity

$$\sin^2(x) + \cos^2(x) = 1$$

obtain the following identities

- $1 + \tan^2(x) = \sec^2(x)$
- $\sec^2(x) - 1 = \tan^2(x)$

We are now going to apply the method of substitution *in reverse*. Recall that the **method of substitution** is an approach to solving the antiderivative problem

$$\int h(x)dx$$

by taking the following strategy:

| |
|---|
| 1. Determine $f(x)$ and $g(x)$ such that, if we write $u = g(x)$ then $h(x) = f(u)\frac{du}{dx}$. |
| 2. Determine $\int f(u)du$ (if possible). |
| 3. Substitute $g(x)$ for $u$. |

Check your understanding

Consider the integral

$$\int(\sin(x) - 1)dx \qquad (**)$$

This is straightforward to solve. Let's make life difficult.

1. By writing $u = \sin(x)$, show that

$$\frac{u-1}{\sqrt{1-u^2}}\frac{du}{dx} = \sin(x) - 1$$

2. If we wanted to use the substitution $u = \sin(x)$ to solve $(**)$ then we would have to determine

$$\int \frac{u-1}{\sqrt{1-u^2}}du$$

   Using the subsitution $v = u^2$, and recalling that $\frac{d}{du}\arcsin(u) = \frac{1}{\sqrt{1-u^2}}$, determine this integral.

**Question:** was the method of substitution really necessary?

**Answer:** ______________

The above example highlights the method of inverse trigonometric substitution. Let's see some examples.

**Example 2.1.** 1. To determine the integral

$$\int \sqrt{9-x^2}dx$$

We make the substitution

$$x = 3\sin(t)$$

Then,

$$\sqrt{9-x^2}\frac{dx}{dt} = 9\cos^2(t)$$

Hence, by inverse substitution

$$\int \sqrt{9-x^2}dx = \int 9\cos^2(t)dt = \underline{\hspace{4cm}}$$

Check your understanding

1. By letting $x = \tan(t)$, use the method of inverse substitution to explain why

$$\int \frac{1}{x^2\sqrt{x^2+1}}dx = \int \frac{\cos(t)}{\sin^2(t)}dt$$

2. Determine

$$\int \frac{\cos(t)}{\sin^2(t)} dt$$