Calculus II: Fall 2017

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 4.5, 7.1.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

The antiderivative problem. Techniques of integration I.

Today we review the antiderivative problem and begin extending our techniques of (indefinite) integration.

1 The antiderivative problem For the next few lectures we are going to investigate the following

> Question: Let $f(x)$ be a function. Does there exist a differentiable function $F(x)$ satisfying
> $$\frac{d}{dx}F(x) = f(x).$$

Check your understanding

Let $f(x) = \log(x)$, where $\log(x)$ is the inverse function of the exponential function $\exp(x)$. Show that $F(x) = x\log(x) - x$ provides a solution to the antiderivative problem for $f(x)$.

If $f(x)$ is a continuous function then the Fundamental Theorem of Calculus provides the general solution to the antiderivative problem:

Fundamental Theorem of Calculus

> Let $f(x)$ be a continuous function defined on the closed interval $a \leq x \leq b$. Then, the function
> $$F(x) = \int_a^x f(u)du$$
> is an antiderivative of $f(x)$.

The Fundamental Theorem of Calculus tells us that any continuous function admits an antiderivative $F(x)$. However, while we are given a recipe for determining an antiderivative $F(x)$ - i.e. 'simply' determine the area below the graph of $f(u)$ as a function of the upper limit of integration $u = x$ - it's not so easy to *know* whether we can recognise $F(x)$.

> QUESTION: Let $f(x)$ be a continuous function. Is there a systematic way to describe the antiderivative $F(x)$ in terms of well-known functions (e.g. rational functions, trigonometric functions, exp, log etc)?

We make precise our notion of a '*well-known*' function.

Definition 1.1. A function $f(x)$ is an **elementary function** if it can be obtained by addition, multiplication, division, and composition from the rational functions (i.e. ratios of polynomial functions), the trigonometric functions and their inverses, and the inverse functions log, exp.

We will spend the next few Lectures introducing a variety of techniques that will help us find solutions to the following problem.

> QUESTION: Let $f(x)$ be an elementary function. Is it possible to describe the antiderivative $F(x)$ as an elementary function?

Remark 1.2.

1. In general, it is not possible to describe the antiderivative of an elementary function as an elementary function. For example, a difficult Theorem states the following:

   $$\text{There is no elementary function } F(x) \text{ such that } \frac{d}{dx}F(x) = \exp(-x^2).$$

   The proof of this result relies on advanced mathematics far beyond the scope of Calculus II (unfortunately).

2. Despite the above remark, the techniques we will develop will allow us to determine antiderivatives of a large family of elementary functions.

3. Recall that the phrase *indefinite integral* is synonymous with *antiderivative*: the indefinite integral

   $$\int f(x)dx$$

   is an expression used in place of '*the (general) antiderivative $F(x)$ of $f(x)$*'. In particular, the problem

   $$\text{Determine } \int f(x)dx$$

   is the same problem as

   $$\text{Find/determine the (general) antiderivative } F(x) \text{ of } f(x).$$

   The Fundamental Theorem of Calculus tells us that we solve this problem using *integration*. As such, an '*indefinite integration problem*' is synonymous with an '*antiderivative problem*'.

2 Integration by parts Recall the Product Rule for derivatives

Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Check your understanding

Take antiderivatives of both sides of the above formula and complete the following expression **Integration by parts**

$$\int fg' = \underline{\hspace{3cm}} - \int \underline{\hspace{3cm}} \qquad (*)$$

Integration by parts provides us with the following strategy to determine the integral $\int h(x)dx$:

Strategy: Find specific functions $f(x)$ and $g(x)$ satisfying

(A) $f(x)g'(x) = h(x)$, and so that

(B) it's easier to determine $\int f'(x)g(x)dx$.

(C) Now, use the integration by parts formula $(*)$.

Example 2.1. 1. Determine

$$\int x\sin(x)dx$$

Try: $f(x) = x$, and $g'(x) = \sin(x)$. Then, $f'(x) = 1$ and $g(x) = -\cos(x)$. Hence, using integration by parts

$$\begin{aligned}\int x\sin(x)dx &= x(-\cos(x)) - \int 1.(-\cos(x))dx \\ &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

Check your understanding

Verify that

$$\frac{d}{dx}(-x\cos(x) + \sin(x)) = x\sin(x).$$

2. Sometimes we may have to integrate by parts several times. For example, consider the following antiderivative problem: Determine

$$\int x^2\exp(x)dx$$

Try: $f(x) = x^2$, $g'(x) = \exp(x)$. Then, $f'(x) = 2x$ and $g(x) = \exp(x)$. Hence,

$$\begin{aligned}\int x^2 \exp(x)dx &= x^2 \exp(x) - \int 2x.\exp(x)dx \\ &= x^2 \exp(x) - \int 2x \exp(x)dx\end{aligned}$$

To determine $\int 2x \exp(x)dx$ we integrate by parts again. **Try:** $f(x) = 2x$, $g'(x) = \exp(x)$. Then, $f'(x) = 2$ and $g(x) = \exp(x)$. Hence,

$$\begin{aligned}\int 2x \exp(x)dx &= 2x \exp(x) - \int 2.\exp(x)dx \\ &= 2x \exp(x) - 2\exp(x) + C\end{aligned}$$

Hence,

$$\int x^2 \exp(x)dx = x^2 \exp(x) - 2x \exp(x) + 2\exp(x) + C$$

(*Recall that the sign of the constant of integration C is irrelevant*)

3. Sometimes we may have to be a bit clever with our choice of $f(x)$ and $g'(x)$. Consider the antiderivative problem: determine

$$\int \log(x)dx$$

Try: $f(x) = \log(x)$, $g'(x) = 1$. Then, $f'(x) = \frac{1}{x}$ and $g(x) = x$. Hence, using integration by parts

$$\begin{aligned}\int \log(x)dx &= \log(x)x - \int \frac{1}{x}.xdx \\ &= \log(x)x - \int 1.dx \\ &= \log(x)x - x + C\end{aligned}$$

4. Sometimes we may want to use integration by parts to find $\int h$ in terms of $\int h$ again, and then solve for $\int h$. Consider the following antiderivative problem: Determine

$$\int \frac{\log(x)}{x}dx$$

Try: $f(x) = \log(x)$, $g'(x) = \frac{1}{x}$. Then, $f'(x) = \frac{1}{x}$ and $g(x) = \log(x)$. Hence,

$$\begin{aligned}\int \frac{\log(x)}{x}dx &= \log(x)\log(x) - \int \frac{1}{x}.\log(x)dx \\ &= (\log(x))^2 - \int \frac{\log(x)}{x}dx \\ \implies 2\int \frac{\log(x)}{x} &= (\log(x))^2\end{aligned}$$

Hence,

$$\int \frac{\log(x)}{x}dx = \frac{1}{2}\left(\log(x)\right)^2$$

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. $\int (x^2 + x)\cos(2x)dx$.
2. $\int x^2 \log(x)dx$, **Try:** $f(x) = \log(x)$, $g'(x) = x^2$.
3. What happens if you take $f(x) = \sin(x)$, $g'(x) = x$ when trying to use integration by parts to determine $\int x\sin(x)dx$?