Middlebury College

Calculus II: Fall 2017

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October 13 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 6.6.
- *Calculus*, Spivak, 3rd Ed.: Section 15.

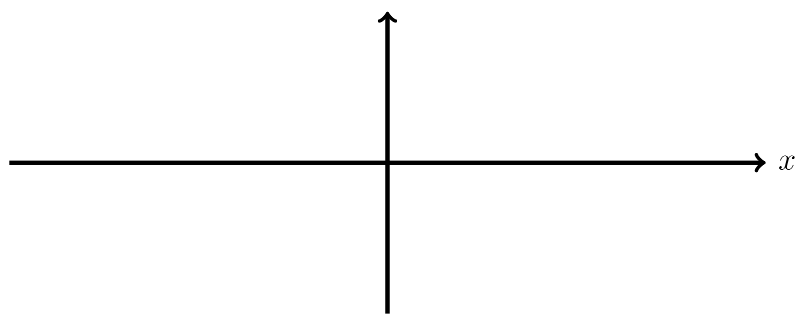
Inverse trigonometric functions

Today we introduce the inverse trigonometric functions and their derivatives.

1 Inverse sine function In this paragraph we will begin an investigation into the *inverse trigonometric functions*

Check your understanding

1. Let $f(x) = \sin(x)$. Draw the graph of $f(x)$.



2. Explain why $f(x)$ is not one-to-one.

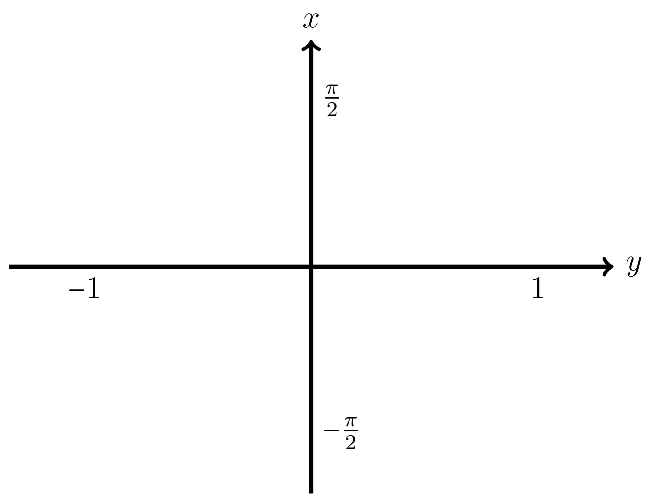
   (Recall: $f(x)$ is one-to-one if *distinct inputs give distinct outputs*)

3. Determine a domain $A : a \leq x \leq b$ on which $f(x)$ is one-to-one.

4. What is the range B of $f(x)$ when the inputs are restricted to A?

5. Explain why an inverse function $f^{-1}(y)$ to $f(x)$ exists, when we restrict to domain A.

6. Draw the graph of $f^{-1}(y)$



Definition 1.1. Consider the function $f(x) = \sin(x)$, with domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Then, $f(x)$ is one-to-one and we call its inverse function $f^{-1}(y)$ the **inverse sine function**, which we denote $\arcsin(y)$.

CHECK YOUR UNDERSTANDING

Complete the following statement:

- the domain of $\arcsin(y)$ is _______________
- the range of $\arcsin(y)$ is _______________

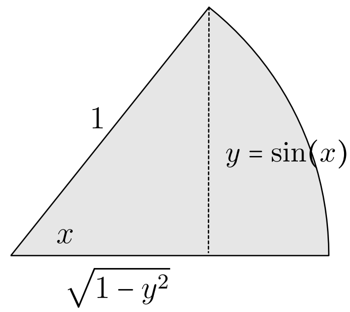
Remark 1.2.
1. We write $\arcsin(y)$ instead of $\sin^{-1}(y)$ to avoid confusion with the common notation $\sin^k(x) = (\sin(x))^k$.
2. As the inverse function of $\sin(x)$, the following functional relationship holds:

   - $\sin(\arcsin(y)) = y$, for every _______________,
   - $\arcsin(\sin(x)) = x$, for every _______________.

3. In words:

   "$\arcsin(y)$ *is the arc whose sine is* y"

   This is demonstrated by the following diagram: (recall that, the length of the arc drawn below is x, whenever the angle x is measured in radians)



Since $f(x) = \sin(x)$ is a differentiable function the same is true of $\arcsin(y)$. Using the formula for the derivative of an inverse function

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

we have

$$\boxed{\frac{d}{dy} \arcsin(y) = \frac{1}{f'(\arcsin(y))} = \frac{1}{\cos(\arcsin(y))} = \frac{1}{\sqrt{1-y^2}}}$$

Here we have used that the derivative of sin is cos, and used the above triangle to show that $\cos(\arcsin(y)) = \sqrt{1-y^2}$.

Hence,

$$\boxed{\arcsin(x) \text{ is an antiderivative of } \tfrac{1}{\sqrt{1-x^2}}}$$

2 Inverse cosine function

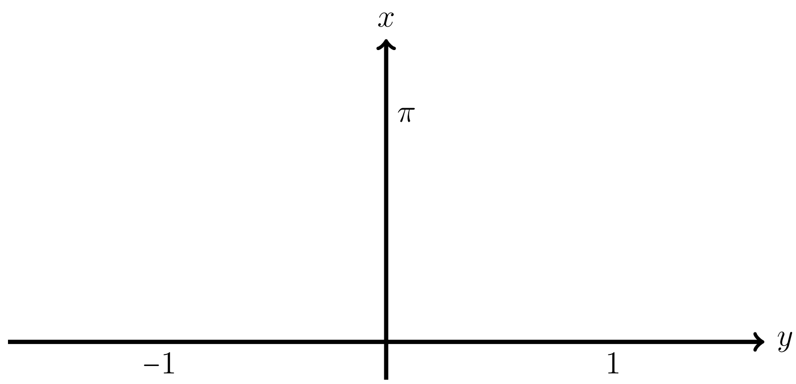
Definition 2.1. Consider the function $f(x) = \cos(x)$, with domain $0 \le x \le \pi$. Then, $f(x)$ is one-to-one and we call its inverse function $f^{-1}(y)$ the **inverse cosine function**, which we denote $\arccos(y)$.

- the domain of $\arccos(y)$ is $-1 \le y \le 1$
- the range of $\arccos(y)$ is $0 \le y \le \pi$

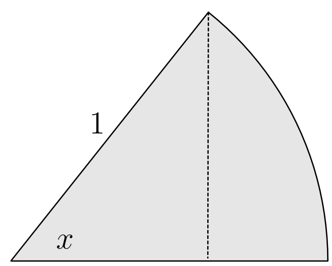
"$\arccos(y)$ *is the arc whose cosine is* y"

Check your understanding

1. Draw the graph of $x = \arccos(y)$



2. Suppose $x = \arccos(y)$. Complete the following diagram as we did above.



3. Deterine the following value:

$$\sin(\arccos(y)) = \underline{\hspace{6em}}$$

4. Use the formula for the derivative of an inverse function Complete the following statement:

arccos(x) is an antiderivative of ____________________

Mathematical workout - flex those muscles

Before Monday's class determine $\arctan'(x)$.