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October 12 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 6.2*, 6.3*.
- *Calculus*, Spivak, 3rd Ed.: Section 18.

The natural logarithm

Today we introduce the *natural logarithm* function as the inverse function of $\exp(x)$.

1 The derivative of inverse functions Let $f(x)$ be a one-to-one function with domain A and range B. Then, f and its inverse function f^{-1} satisfy the following functional relationship:

$$f(f^{-1}(y)) = y, \quad \text{for every } y \text{ in } B, \tag{$*$}$$

$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in } A.$$

If $f(x)$ is also a differentiable function (i.e. its derivative $f'(x)$ exists for every x in its domain A) then its inverse function is also differentiable. In fact, the derivative of $f^{-1}(y)$ can be determined in terms of the derivative of $f'(x)$.

First, we recall the *chain rule* from Calculus I.

Check your understanding

Compute $\frac{dy}{dx}$ where

$$y = x^2 + 4x + \frac{1}{x^2 + 4x}$$

The precise version of the Chain Rule is given as follows:

Chain Rule

Let f and g be differentiable functions. Then,

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Example 1.1. Let $f(x) = x + \frac{1}{x}$ and $g(x) = x^2 + 4x$. We have

$$f'(x) = 1 - \frac{1}{x^2}, \qquad g'(x) = 2x + 4.$$

Then, the chain rule states that

$$\frac{d}{dx}\left(x + \frac{1}{x^2 + 4x}\right) = \frac{d}{dx} f(g(x)) = f'(x^2 + 4x) \cdot g'(x) = \left(1 - \frac{1}{(x^2 + 4x)^2}\right)(2x + 4)$$

You can check that this agrees with your calculation above.

Now we investigate a useful expression for the differentiability of an inverse function.

Check your understanding

Recall the function $f(x) = 1 - \frac{1}{x}$ from October 11 Lecture. We determined the inverse function to be

$$f^{-1}(y) = \frac{1}{1 - y}$$

1. Compute

$$\frac{d}{dy} f^{-1}(y).$$

2. Compute $f'(x)$.

3. Show that

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}.$$

Proposition 1.2. *Let $f(x)$ be a differentiable one-to-one function. Suppose that $f'(f^{-1}(y)) \neq 0$, for all y. Then, $f^{-1}(y)$ is differentiable and*

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

Proof: This follows from the functional relationship $(*)$ and the chain rule. We have, for every y,

$$f(f^{-1}(y)) = y.$$

Now, differentiating with respect to y (remember, we are wanting the derivative of the function $f^{-1}(y)$ with respect to y) and using the chain rule, we find

$$1 = \frac{d}{dy}\left(f(f^{-1}(y))\right) = f'(f^{-1}(y)) \cdot (f^{-1})'(y)$$

$$\implies \quad \frac{d}{dy}\left(f^{-1}(y)\right) = (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

□

2 The natural logarithm I We apply Proposition 1.2 to the function $f(x) = \exp(x)$.

First, we recall the following facts about $\exp(x)$:

1. $\exp(x)$ is strictly increasing. Hence, $\exp(x)$ is one-to-one.
2. The domain of $\exp(x)$ is the collection of all real numbers.
3. The range of $\exp(x)$ is the collection of all $y > 0$: indeed, since $\exp(x)$ is differentiable it is continuous. We have also seen that $\exp(x) \to +\infty$ as $x \to +\infty$. Using $\exp(-x) = 1/\exp(x)$ we see that $\exp(x) \to 0$ as $x \to -\infty$. This implies that the range of $\exp(x)$ is the collection of all positive real numbers.

Hence, $\exp(x)$ **has an inverse function** $\exp^{-1}(x)$.

Remark 2.1. 1. By the above remarks,

- the domain of $\exp^{-1}(y)$ is the collection of all $y > 0$
- the range of $\exp^{-1}(y)$ is the collection of all real numbers

2. We will often write $\exp^{-1}(x)$ instead of $\exp^{-1}(y)$. Remember, it doesn't matter what symbol we use for our input variable as long as we are consistent in a computation.

We have seen that $f(x) = \exp(x)$ is a differentiable function so that $\exp^{-1}(y)$ is also differentiable. We apply Proposition 1.2 to obtain

$$\frac{d}{dy} \exp^{-1}(y) = \frac{1}{f'(\exp^{-1}(y))}$$

Recall that $f'(x) = \exp(x)$. Therefore, $f'(\exp^{-1}(y)) = \exp(\exp^{-1}(y)) = y$, using functional property $(*)$. Hence,

$$\frac{d}{dy} \exp^{-1}(y) = \frac{1}{y} \qquad (**)$$

3 A Fundamental interlude Let $f(x)$ be a function. An **antiderivative** of $f(x)$ is a differentiable function $F(x)$ satisfying

$$\frac{d}{dx}F(x) = f(x).$$

Proposition 3.1. *If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ then*

$$F(x) = G(x) + c,$$

for some constant c. In particular, if $F(x)$ is some antiderivative of $f(x)$ then every antiderivative of $f(x)$ is of the form

$$F(x) + c, \qquad \textit{where } c \textit{ is a constant.}$$

Remark 3.2. This Proposition indicates where the *constant of integration* comes from when we are computing antiderivatives: given a function $f(x)$ there is a family of antiderivatives associated to $f(x)$.

The most important Theorem you saw in Calculus I was an approach to determining the antiderivative of a continuous function.

Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function defined on the closed interval $a \leq x \leq b$. Then, the function

$$F(x) = \int_a^x f(u)du$$

is an antiderivative of $f(x)$.

4 The natural logarithm II We can restate $(**)$ as follows:

$\exp^{-1}(x)$ *is an antiderivative of* $g(x) = \frac{1}{x}$.

We now give a name to a particular antiderivative of $g(x) = \frac{1}{x}$.

Definition 4.1. The **natural logarithm function** is the function

$$\log(x) = \int_1^x \frac{dt}{t}$$

By the Fundamental Theorem of Calculus, $\log(x)$ is an antiderivative of $\mathfrak{g}(x) = \frac{1}{x}$. Hence, Proposition 3.1 implies that there is a constant c so that

$$\exp^{-1}(x) = \log(x) + c.$$

Since $\exp(0) = 1$, we must have $\exp^{-1}(1) = 0$, giving

$$0 = \exp^{-1}(1) = \log(1) + c = \int_1^1 \frac{dt}{t} + c = c.$$

Hence,

The natural logarithm function $\log(x)$ defined above *is* the inverse function $\exp^{-1}(x)$,

$$\log(x) = \exp^{-1}(x)$$

Remark 4.2.

1. In Problem Set 4 you will show that the function $\log(x)$ just defined as the inverse of $\exp(x)$ satisfies the expected *logarithm rules*:

   - $\log(xy) = \log(x) + \log(y)$, for every $x, y > 0$.
   - $\log(x^n) = n\log(x)$, for every $x > 0$ and natural number n.

2. As the inverse function of $\exp(x)$, the following functional relationships hold:

   - $\exp(\log(y)) = y, \quad$ for every $y > 0$,
   - $\log(\exp(x)) = x, \quad$ for every x.

3. We have claimed that

   $$\exp(x) = e^x, \qquad \text{where } e = \exp(1) \text{ is Euler's number.}$$

   In this way we see that $\log(x) = \ln(x)$ is the *logarithm base* e function.

Check your understanding

1. Use the definition $\log(x) = \int_1^x \frac{dt}{t}$ to explain why log is an increasing function: i.e. if $0 < x < y$ explain why $\log(x) < \log(y)$.

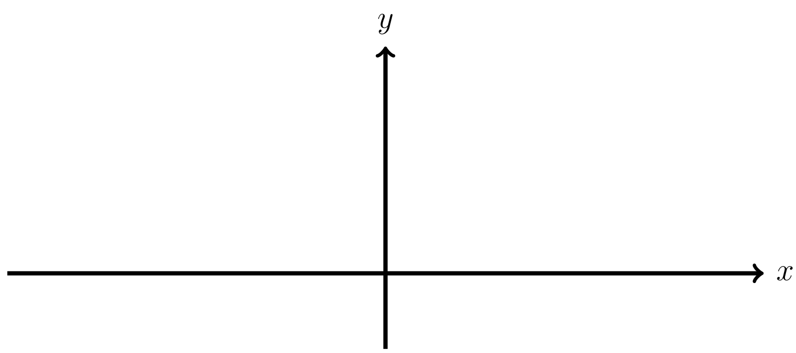
2. Using the fact that $\log(x)$ is the inverse function to $\exp(x)$, complete the following statements:

   - $\log(x) > 0$ whenever __________
   - $\log(x) < 0$ whenever __________

5 Inverse trigonometric functions In this paragraph we will begin an investigation into the *inverse trigonometric functions*

CHECK YOUR UNDERSTANDING

1. Let $f(x) = \sin(x)$. Draw the graph of $f(x)$.



2. Explain why $f(x)$ is not one-to-one.

3. Determine a domain $A:\ a \leq x \leq b$ on which $f(x)$ is one-to-one.

4. What is the range B of $f(x)$ when the inputs are restricted to A?

5. Explain why an inverse function $f^{-1}(y)$ to $f(x)$, when we restrict to domain A, exists.

6. Draw the graph of $f^{-1}(y)$