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October 11 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 6.1, 6.2*, 6.3*.
- *Calculus*, Spivak, 3rd Ed.: Section 18.
- *AP Calculus BC*, Khan Academy:

Inverse functions. The natural logarithm.

Today we introduce the notion of an *inverse function*. We will see that the fact that $\exp(x)$ is *strictly increasing* means the equation $c = \exp(x)$, where $c > 0$ is a constant, has a unique solution. Determining this unique solution will require us to recall, in our next lecture, the *Fundamental Theorem of Calculus*.

1 One-to-one functions Recall the function

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

You have seen, and determined most of, the following properties of $\exp(x)$:

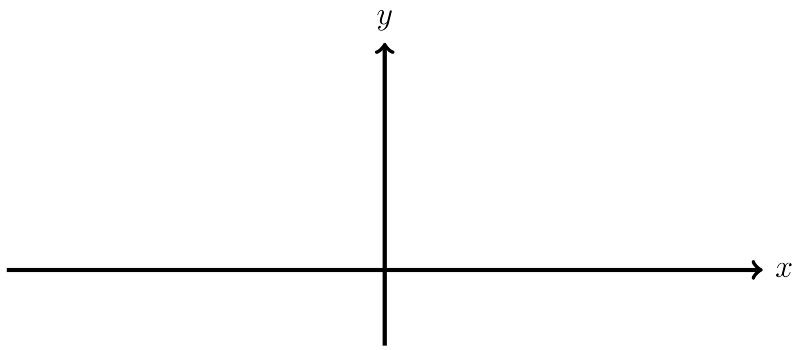
- $\exp(x) > 1$, for any real number $x > 0$.
- $\exp(0) = 1$.
- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) > 0$, for any real number x.
- $\exp(x + y) = \exp(x) \cdot \exp(y)$, for any real numbers x, y. $(*)$
- $\exp(x)$ is **strictly increasing**.
- $\exp(x) = e^x$, where $e = \exp(1)$ is **Euler's number**.
- $\exp(x)$ is differentiable, for every x, and its derivative is itself

$$\frac{d}{dx}\exp(x) = \exp(x).$$

We call property $(*)$ the **Remarkable Property of** $\exp(x)$.

Check your understanding

Draw the graph of the function $\exp(x)$.



Let $c > 0$ be your favourite positive real number. Explain why the equation

$$c = \exp(x)$$

has a unique solution.

Definition 1.1. Let $f(x)$ be a function. We say that $f(x)$ is **one-to-one** if $f(x) \neq f(y)$ whenever $x \neq y$ lie in the domain of f. In words,

$f(x)$ is one-to-one if it never takes on the same output value twice.

Example 1.2.

1. The function $f(x) = x^2$, with domain being the collection of all real numbers, is not one-to-one as $f(-1) = 1 = f(1)$.
2. The function $g(x) = x^3$, with domain being the collection of all real numbers, is one-to-one: first you can check that
$$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$
Now, if $x \neq y$ then the right hand side of the above expression can only be equal to 0 whenever
$$x^2 + xy + y^2 = 0.$$
By completing the square we find
$$x^2 + xy + y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{3}{4}y^2 \geq 0$$
Thus,
$$0 = x^2 + xy + y^2 \quad \Leftrightarrow \quad x + \frac{y}{2} = 0 \text{ and } y = 0 \quad \Leftrightarrow \quad x = y = 0,$$
contradicting our assumption that $x \neq y$. Hence, $x^2 + xy + y^2 > 0$ whenever $x \neq y$, so that $x^3 \neq y^3$ whenever $x \neq y$.

3. The function $h(x) = x^2$, with domain being the collection of all real numbers $x \geq 0$, is one-to-one.

We have the following useful test to determine when a function is one-to-one:

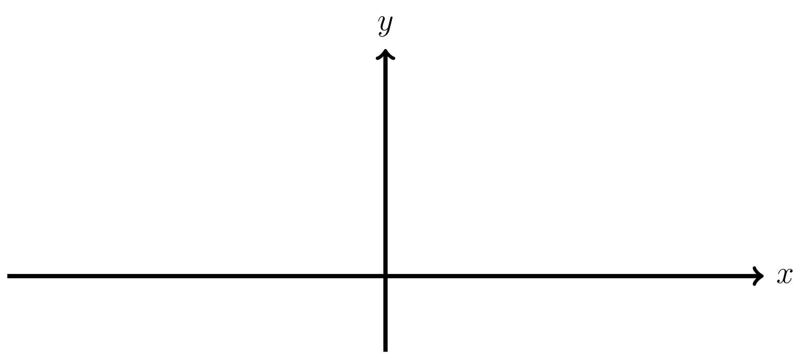
Horizontal Line Test

> $f(x)$ is one-to-one if every horizontal line intersects its graph at most once.

Check your understanding

1. Let $f(x) = \exp(x)$ be the exponential function. Explain why $\exp(x)$ is one-to-one.

2. Suppose that $f(x)$ is a strictly decreasing function. Draw a representative graph of the function below



3. Complete the following statement:

> If a function $f(x)$ is ________________________________ then $f(x)$ is one-to-one.

2 Inverse functions In this paragraph we will introduce the notion of an inverse function. You have determined above the following result.

Proposition 2.1. *Let $f(x)$ be a strictly increasing/decreasing function and let c be a real number. Then, the equation*

$$c = f(x)$$

has <u>at most one solution</u>.

Example 2.2. 1. Let $f(x) = x^3$, with domain being the collection of all real numbers x, and $c = -27$. Then, $x = -3$ is a solution to the equation $-27 = x^3$. Moreover, this is the only solution.

2. Let $f(x) = 1 - \frac{1}{x}$, with domain being the collection of real numbers $x > 0$. Let $c = -1$. Then, the equation

$$-1 = 1 - \frac{1}{x}$$

has the unique solution $x = \frac{1}{2}$.

However, if $c = 3$ then there is no $x > 0$ such that

$$3 = 1 - \frac{1}{x}.$$

Definition 2.3. Let $f(x)$ be a function. The **range of** $f(x)$ is the collection of all outputs of $f(x)$.

Example 2.4. 1. Let $f(x) = x^3$ with domain being the collection of all real numbers x. Then, the range of $f(x)$ is the collection of all real numbers: if c is any real number then $c = \left(\sqrt[3]{c}\right)^3 = f\left(\sqrt[3]{c}\right)$. That is, c is the output of some input for the function $f(x) = x^3$.

2. Let $f(x) = 1 - \frac{1}{x}$ with domain being the collection of all real numbers $x > 0$. Then, the range of $f(x)$ is the collection of all real numbers $c < 1$: if $c < 1$ then $x = \frac{1}{1-c} > 0$ satisfies $f(x) = c$.

We are now ready to introduce our main definition.

Definition 2.5. Let $f(x)$ be a one-to-one function with domain A and range B. Then, its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad y = f(x).$$

In words,

If $y = f(x)$ is an output of f then y is an input of f^{-1} and $f^{-1}(y) = x$.

Example 2.6. 1. Let $f(x) = 1 - \frac{1}{x}$ with domain A being the collection of all real numbers $x > 0$. The range of f, B, is the collection of all real numbers $y < 1$. The function $f(x)$ is one-to-one and its inverse function is

$$f^{-1}(y) = \frac{1}{1-y}.$$

Remark 2.7. 1. To determine the inverse function f^{-1} of a one-to-one function f, solve the equation

$$y = f(x)$$

for x in terms of y.

2. It is important to remember that $f^{-1}(y) \neq \frac{1}{f(y)}$, in general. For example, if $f(x) = 1 - \frac{1}{x}$ then

$$\frac{1}{f(y)} = \frac{y}{y-1} \neq \frac{1}{1-y} = f^{-1}(y)$$

3. Let $f(x)$ be a one-to-one function with domain A and range B. Then, f and its inverse function f^{-1} satisfy the following functional relationship:

$$f(f^{-1}(y)) = y, \quad \text{for every } y \text{ in } B,$$

$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in } A.$$

If $f(x)$ is a differentiable one-to-one function (i.e. its derivative $f'(x)$ exists for every x in its domain) then its inverse function is also differentiable.

Proposition 2.8. *Let $f(x)$ be a differentiable one-to-one function. Suppose that $f'(f^{-1}(y)) \neq 0$, for all y. Then, $f^{-1}(y)$ is differentiable and*

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

Proof: This follows from the functional relationship described in the above Remark and the chain rule. We have, for every y,

$$f(f^{-1}(y)) = y.$$

Now, differentiating with respect to y and using the chain rule, we find

$$1 = \frac{d}{dy}\left(f(f^{-1}(y))\right) = f'(f^{-1}(y)) \cdot (f^{-1})'(y)$$

$$\implies \quad \frac{d}{dy}\left(f^{-1}(y)\right) = (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

□