Calculus II: Fall 2017

Contact: gmelvin@middlebury.edu

November 9 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.1, 9.3.
- *Calculus II*, Marsden, Weinstein: Chapter 11.3.
- *AP Calculus BC*, Khan Academy: Improper Integrals.

Differential Equations I

In this lecture we will begin an investigation of differential equations. These are quite different to the equations you have seen in the past in that solutions are *functions*, and not numbers. We discuss what differential equations are, what a solution to a differential equation is, how differential equations arise in nature and investigate a class of differential equations known as *separable equations*.

1 What is a differential equation? Differential equations are equations involving the derivative or derivatives of an unknown function $f(x)$. For example,

$$\frac{df}{dx} = x\sin(x), \qquad (f'(x))^2 + f''(x) = x + 1, \qquad \frac{d^2f}{dx^2} = -4f(x)$$

are all examples of differential equations.

A **solution** of a differential equation is a function $f(x)$ whose derivatives satisfy the given equation.

For example, the function $f(x) = \cos(2x)$ is a solution to the differential equation

$$\frac{d^2f}{dx^2} = -4f(x)$$

Indeed, we have

$$\frac{d^2f}{dx^2} = -4\cos(4x) = -4f(x).$$

Aim: given a differential equation, find all solutions $f(x)$.

Remark 1.1. Recall the (Leibniz) notation:

$$\frac{d}{dx}f(x) = \frac{df}{dx} = f'(x)$$

More generally,

$$\frac{d^nf}{dx^n} = \frac{d^n}{dx^n}f(x)$$

denotes the n^{th} derivative of $f(x)$.

Example 1.2. 1. To **solve** the differential equation

$$\frac{df}{dx} = x\sin(x)$$

we seek *all* functions $f(x)$ which satisfy

$$\frac{d}{dx} f(x) = x\sin(x).$$

CHECK YOUR UNDERSTANDING

(a) Verify that $f(x) = -x\cos(x) + \sin(x) - 5$ is a solution to the above differential equation.

(b) Express the differential equation as an antiderivative problem to complete the following statement:

Find $f(x)$ such that

$$f(x) = \underline{\hspace{5cm}}$$

(c) Find all possible solutions to the differential equation

$$\frac{d}{dx} f(x) = x\sin(x)$$

2. Consider the differential equation

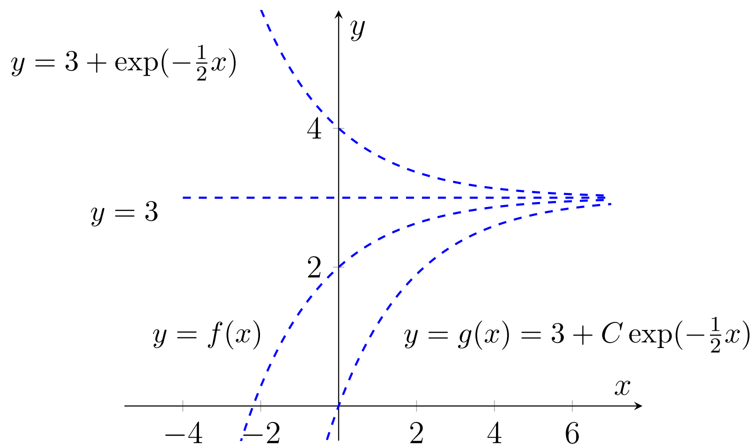
$$\frac{df}{dx} = \frac{3}{2} - \frac{1}{2} f(x)$$

CHECK YOUR UNDERSTANDING

(a) Which of the following functions $f(x)$ are solutions to the differential equation?

- $f(x) = \frac{3}{2}x - \frac{1}{2}x^2$
- $f(x) = 3$
- $f(x) = 3 - 5\exp(-\frac{1}{2}x)$
- $f(x) = 3 + C\exp(-\frac{1}{2}x)$, for any constant C

(b) The graphs of four solutions to the differential equation are given below:



Determine a such that the solution $f(x)$ satisfies $f(0) = a$

Determine C such $g(0) = 0$.

2 Growth & Decay Differential equations arise EVERYWHERE in the natural world. A differential equation prescribes restrictions on the behaviour of the deriviatives of a function $f(x)$. You have learned in your previous calculus courses that the deriviatives of a function $f(x)$ provide various measures of the *rate of change* of $f(x)$. Therefore, it should not be surprising to learn that any situation where quantities can change (with respect to time, say) can be described by differential equations.

We will now investigate the fundamental examples of **growth** and **decay**.

Population Growth & Decline Let

$$P(t) = \text{ the number of individuals in a population of interest at time } t$$

Here t is the independent *time* variable. For example, the population of interest could be

- population of persons infected with a pandemic flu(!)
- population of bees in the Serengeti
- population of carbon dioxide molecules in the atmosphere
- population of dollars in your bank account

Suppose a population $P(t)$ **grows** at an annual rate of 3%, say, where t is the time in years. Then, we can formulate this mathematically by utilising the quantity $\frac{dP}{dt}$ to denote the rate of change (i.e. rate of growth). Therefore, if $P(t)$ is growing at a rate of 3% per annum, then it should seem reasonable to represent this relationship with the equation

$$\frac{dP}{dt} = 0.03P.$$

Aim: describe $P(t)$.

CHECK YOUR UNDERSTANDING

1. Check that $f(t) = \exp(0.03t)$ is a solution to the differential equation.

2. Determine two further distinct solutions $g(t), h(t)$ to this differential equation

3. Give a candidate for a general solution to this differential equation.

GET CREATIVE! Suppose that $P(t)$ describes a population in decline (for example, $P(t)$ could model the decay of a radioactive substance): say, $P(t)$ is decreasing by $\frac{1}{2}$ every day.

1. Write down a differential equation modelling the behaviour of $P(t)$.

2. Can you determine a solution to this differential equation? If so, write it down.

In general, we have the following

Growth & Decay Equation

Let $P(t)$ be the number of individuals in a population of interest at time t. Then, the basic growth/decay equation governing $P(t)$ is

$$\frac{dP}{dt} = kP$$

- If $k > 0$ then $P(t)$ is **growing**.
- If $k < 0$ then $P(t)$ is **decaying**.

Remark 2.1. The growth & decay equation given above describes the growth/decay of populations, *in the absence of external factors*. As such, this equation does not perfectly model most growth/decay situations.

For example, suppose that a population initially grows at a constant rate proportional to $P(t)$, but begins to decay once $P(t)$ passes a certain threshold M (the *carrying capacity of* $P(t)$)

- $\frac{dP}{dt} = kP$ if P is small.
- $\frac{dP}{dt} < 0$ if $P > M$ (i.e. P decreases once it exceeds M).

A simple expression that incorporates both assumptions is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad k > 0.$$

Indeed, if P is small compared to M then P/M is small, while if $P > M$ then $1 - \frac{P}{M} < 0$ so that $\frac{dP}{dt} < 0$.