Middlebury College

# Calculus II: Fall 2017

**Contact:** `gmelvin@middlebury.edu`

# November 8 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.8.
- *Calculus II*, Marsden, Weinstein: Chapter 11.3.
- *AP Calculus BC*, Khan Academy: Improper Integrals.

## Improper integrals II

In this lecture we will complete our discussion of improper integrals. We will investigate how we may define integrals for functions admitting infinite discontinuities. Such integrals will be called *type II improper integrals*.

**1 The Improper Integral Comparison Test** Yesterday we defined improper integrals - an improper integral is an expression of the form

$$\boxed{\int_0^\infty f(x)dx, \qquad \int_{-\infty}^0 f(x)dx, \qquad \int_{-\infty}^\infty f(x)dx}$$

**Remark 1.1.** Recall that an improper integral is not a integral in the usual sense (i.e. it is not defined as the limit of Riemann sums). For example,

$$\int_0^\infty f(x)dx \stackrel{def}{=} \lim_{t\to\infty}\int_0^t f(x)dx$$

We introduce the following important family of improper integrals.

**Example 1.2.** Let $r$ be a real number. Let $f(x) = \frac{1}{x^r}$. Then,

$$\int f(x)dx = \begin{cases} \frac{1}{1-r}\frac{1}{x^{r-1}}, & r \neq 1, \\ \log(x), & r = 1. \end{cases}$$

Hence, for any $a > 1$,

$$\int_1^a f(x)dx = \begin{cases} \frac{1}{1-r}\left(a^{1-r} - 1\right), & r \neq 1 \\ \log(a), & r = 1 \end{cases}$$

Observe,

$$\lim_{a\to\infty}\frac{1}{1-r}\left(a^{1-r}-1\right) = \begin{cases} \underline{\hspace{6cm}}, & \text{if } r > 1, \\ \\ \underline{\hspace{6cm}}, & \text{if } r < 1. \end{cases}$$

Let $r$ be a real number. The improper integral

$$\int_1^\infty \frac{1}{x^r} dx$$

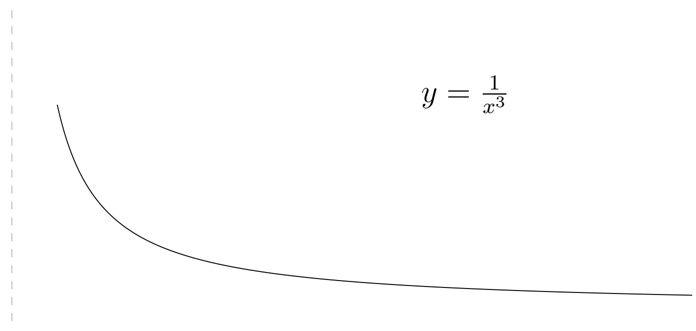
- **is convergent** if $r > 1$,
- **is divergent** if $r \leq 1$.

CHECK YOUR UNDERSTANDING

1. Which of the following improper integrals are convergent?

| | | |
|---|---|---|
| • $\int_1^\infty \frac{1}{x^{10}} dx$ | CONVERGENT | DIVERGENT |
| • $\int_1^\infty \frac{1}{x} dx$ | CONVERGENT | DIVERGENT |
| • $\int_1^\infty x^{1/3} dx$ | CONVERGENT | DIVERGENT |
| • $\int_1^\infty x^{-4/3} dx$ | CONVERGENT | DIVERGENT |

2. Below is drawn the graph $y = \frac{1}{x^3}$, where $x \geq 1$. Draw the graph of $f(x) = \frac{1}{x^3+x+1}$, where $x \geq 1$.




3. Use the previous problem to deduce the convergence/divergence of the improper integral

$$\int_1^\infty \frac{1}{x^3 + x + 1} dx$$

CONVERGENT DIVERGENT

4. Explain your answer by comparing $\int_1^\infty f(x) dx$ with $\int_1^\infty x^{-3} dx$.

Get creative!

Complete the following statements to deduce the

**Improper Integral Comparison Test (IICT)**:

Let $f(x) \geq g(x) \geq 0$ be continuous functions defined on $[a, \infty)$. Then,

1.

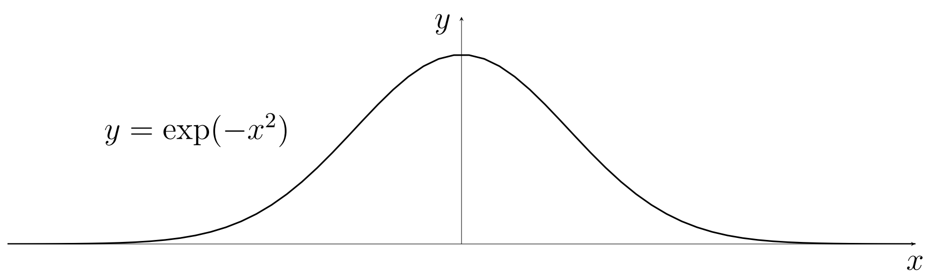
$$\int_a^\infty f(x)dx \quad \underline{\hspace{3cm}} \quad \Longrightarrow \quad \int_a^\infty g(x)dx \quad \underline{\hspace{3cm}}$$

2.

$$\int_a^\infty g(x)dx \quad \underline{\hspace{3cm}} \quad \Longrightarrow \quad \int_a^\infty f(x)dx \quad \underline{\hspace{3cm}}$$

**Example 1.3.** Consider the improper integral

$$\int_1^\infty \exp(-x^2)dx$$




It is a Theorem that any antiderivative of $f(x) = \exp(-x^2)$ as not an elementary function. This means it's hard(!) for us to determine the convergence of the above improper integral: we are unable to write down an antiderivative in terms of functions we recognise.

Let's see how the IICT helps us: for any $x \geq 1$, we have

$$x \leq x^2 \quad \Longrightarrow \quad -x^2 \leq -x$$

As $\exp(x)$ is an increasing function this gives $\exp(-x^2) \leq \exp(-x)$, whenever $x \geq 1$. Now,

$$\int_1^a \exp(-x)dx = [-\exp(-x)]_1^a = \exp(-1) - \exp(-a) \ \to \ \exp(-1) = e^{-1} \text{ as } a \to \infty.$$

Hence,

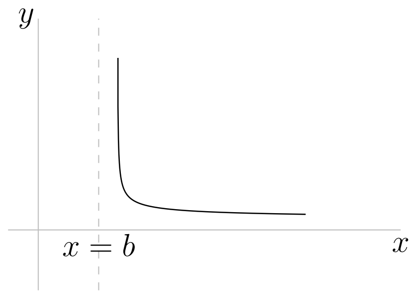
$$\int_1^\infty \exp(-x)dx \text{ convergent} \quad \Longrightarrow \quad \int_1^\infty \exp(-x^2)dx \text{ convergent.}$$

**2 Type II Improper Integrals** We will consider how to approach determining the area under the graph of a a function that admits infinite discontinuities.

RECALL: Let $f(x)$ be a nonnegative function, continuous on $[a, b)$ or $(b, a]$ and suppose $\lim_{x \to b} f(x) = +\infty$. Then, $x = b$ is called an **infinite discontinuity of** $f(x)$.

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES!

Consider the function $f(x) = \frac{1}{\sqrt{x-2}}$. A portion of the graph of $f(x)$ is shown below




1. Determine $b$ so that $f(x)$ admits an infinite discontinuity at $x = b$.

2. Let $a$ be a real number so that $b < a < 5$. Determine

$$\int_a^5 \frac{1}{\sqrt{x-2}} dx$$

3. Is the area between the graph of $f(x)$ and the $x$-axis finite or infinite? If finite, what is the area? If infinite, explain why.

The above investigation leads us to the following definition.

**Type II Improper Integrals**

Let $f(x)$ be a nonnegative function. Suppose that $x = b$ is an infinite discontinuity of $f(x)$.

- Suppose $f(x)$ is continuous on $[a, b)$. If $\lim_{t\to b} \int_a^t f(x)dx$ exists (and is finite) then we define

$$\int_a^b f(x)dx \stackrel{def}{=} \lim_{t\to b} \int_a^t f(x)dx$$

- Suppose $f(x)$ is continuous on $(b, a]$. If $\lim_{t\to b} \int_t^a f(x)dx$ exists (and is finite) then we define

$$\int_b^a f(x)dx \stackrel{def}{=} \lim_{t\to b} \int_t^b f(x)dx$$

In either case, we say that $\int_a^\infty f(x)dx$ (resp. $\int_{-\infty}^b f(x)dx$) is a **convergent (improper) integral**. Otherwise, the (improper) integral is **divergent**.

**Remark 2.1.** An integral $\int_a^b f(x)dx$ defined over an interval $[a, b]$ on which $f(x)$ admits an infinite discontinuity is called a **type II improper integral**. It is not an integral in the usual sense (i.e. it is not defined as the limit of Riemann sums).

**Example 2.2.** 1. Consider the improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$$

Since the integrand $\frac{1}{\sqrt{1-x^2}}$ admits an infinite discontinuity at $x = 1$ the integral is a type II improper integral. Hence, by definition

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to 1} \int_0^t \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to 1} \arcsin(t) = \frac{\pi}{2}.$$

Hence, the improper integral is convergent.

2. Consider the function $f(x) = \frac{1}{x-2}$. There exists an infinite discontinuity of $f(x)$ at $x = 2$. Then, the integral

$$\int_0^2 \frac{1}{x-2}dx$$

is improper. Moreover, by definition

$$\int_2^5 \frac{1}{x-2}dx = \lim_{t\to 2} \int_2^5 \frac{1}{x-2}dx = \lim_{t\to 2} [\log(x-2)]_t^5 = \lim_{t\to 2} (\log(3) - \log(t-2)) = +\infty$$

Hence, the improper integral is divergent.

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

Determine whether the given improper integral is convergent or divergent.

1. $$\int_1^\infty \frac{1}{(2x+1)^3}dx$$

2. $$\int_1^{\sqrt{2}} \frac{1}{x^2-1}dx$$