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November 6 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.8.
- *Calculus II*, Marsden, Weinstein: Chapter 11.3.
- *AP Calculus BC*, Khan Academy: Improper Integrals.

Improper integrals I

In this lecture we will investigate what we could mean by an integral on an *unbounded domain*. We will define *improper integrals of (type I)* and give some examples.

1 Unbounded definite integrals Given any continuous function $f(x)$, defined on the closed interval $[a, b]$, we *define*

$\int_a^b f(x)dx$	$=$	(signed) area lying between graph $y = f(x)$ and x-axis, for $a \leq x \leq b$

In particular, definite integrals have only been defined on **bounded intervals**.

> Question: how might we define definite integrals on **unbounded intervals**?

Remark 1.1. We can't mimic the bounded interval case and use Riemann sums: this requires us to subdivide an interval $[a, b]$ into n subintervals and consider the limit of sums of areas of rectangles. In the unbounded case we would necessarily be forced to consider rectangles having 'infinite' base!

Mathematical workout - flex those muscles

Consider the function $f(x) = \frac{1}{x^2}$. For a natural number n define

$$a_n = \int_1^n f(x)dx$$

1. Evaluate a_1, a_2, a_{10}.

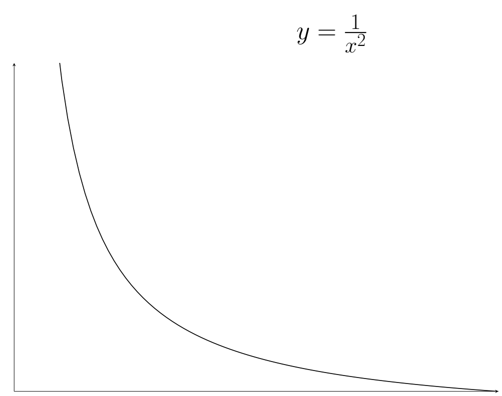
2. Spot the pattern! Determine the general expression:

$$a_n = \underline{\hspace{3cm}}$$

3. Circle the correct description.

- (a_n) is BOUNDED UNBOUNDED
- (a_n) is MONOTONIC NON-MONOTONIC

4. Explain your second choice above using the graph.



5. Is the sequence (a_n) convergent? If so, determine $L = \lim_{n\to\infty} a_n$. If not, justify.

6. Is the area under the (infinite) graph $y = \frac{1}{x^2}$, where $x \geq 1$, finite or infinite? If finite, determine this area. If infinite, explain why.

2 Type I Improper Integrals

- Let $f(x)$ be a nonnegative function, continuous on the interval $[a, \infty)$. If $\lim_{t\to\infty} \int_a^t f(x)dx$ exists (and is finite) then we define

$$\int_a^\infty f(x)dx \overset{def}{=} \lim_{t\to\infty} \int_a^t f(x)dx$$

- Let $f(x)$ be a nonnegative function, continuous on the interval $(-\infty, b]$. If $\lim_{t\to-\infty} \int_t^b f(x)dx$ exists (and is finite) then we define

$$\int_{-\infty}^b f(x)dx \overset{def}{=} \lim_{t\to-\infty} \int_t^b f(x)dx$$

In either case, we say that $\int_a^\infty f(x)dx$ (resp. $\int_{-\infty}^b f(x)dx$) is a **convergent (improper) integral**. Otherwise, the (improper) integral is **divergent**.

- An (improper) integral

$$\int_{-\infty}^\infty f(x)dx \overset{def}{=} \int_0^\infty f(x)dx) + \int_{-\infty}^0 f(x)dx$$

is **convergent** if and only if *both the integrals* on the right hand side are convergent. In particular, the improper integral is **divergent** if either of the improper integrals on the right hand side are divergent.

Remark 2.1. 1. An integral (irrespective of convergence) of the form

$$\int_a^\infty f(x)dx \qquad \text{or} \qquad \int_{-\infty}^b f(x)dx \qquad \text{or} \qquad \int_{-\infty}^\infty f(x)dx$$

is called an **improper integral**. Improper integrals are not 'proper' because they are not obtained as a limit of Riemann sums. This is important to remember.

2. Improper integrals can also be defined for arbitrary (i.e. not necessarily nonnegative) functions. However, we will only consider the nonegative situation.

Example 2.2. 1. Consider the improper integral

$$\int_{-\infty}^\infty \frac{1}{x^2+1}dx$$

The integrand is unchanged under the change of variable $x \leftrightarrow -x$, so that

$$\int_0^a \frac{1}{x^2+1}dx = \underline{\hspace{4cm}}, \qquad \text{for any } a.$$

Hence, $\int_0^\infty \frac{1}{x^2+1}dx$ is convergent if and only if $\int_{-\infty}^0 \frac{1}{x^2+1}dx$ is convergent. If either (hence, both) of these integrals are convergent then

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx = \underline{\hspace{4cm}}$$

We compute

$$\int_0^a \frac{1}{x^2+1}dx = [\arctan(x)]_0^a = \arctan(a)$$

Now,

$$t = \arctan(a) \quad \Leftrightarrow \quad \tan(t) = a$$

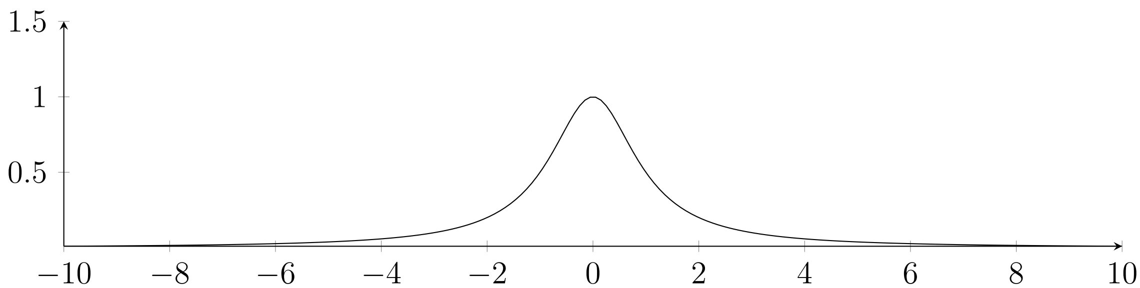
Hence, as $a \to \infty$,

$$t = \arctan(a) \longrightarrow \underline{\hspace{2cm}}$$

Hence, the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$$

is $\underline{\hspace{3cm}}$.



2. Let r be a real number. Let $f(x) = \frac{1}{x^r}$. Then,

$$\int f(x)dx = \begin{cases} \frac{1}{1-r}\frac{1}{x^{r-1}}, & r \neq 1, \\ \log(x), & r = 1. \end{cases}$$

Hence, for any $a > 1$,

$$\int_1^a \frac{1}{x^r}dx = \begin{cases} \frac{1}{1-r}\left(a^{1-r} - 1\right), & r \neq 1 \\ \log(a), & r = 1 \end{cases}$$

Observe,

$$\lim_{a\to\infty} \frac{1}{1-r}\left(a^{1-r} - 1\right) = \begin{cases} -\frac{1}{1-r}, & \text{if } r > 1, \\ +\infty, & \text{if } r < 1. \end{cases}$$

Let r be a real number. The improper integral

$$\int_1^\infty \frac{1}{x^r}dx$$

- **is convergent** if $r > 1$,
- **is divergent** if $r \leq 1$.

Check your understanding

1. Determine the convergence of the improper integral

$$\int_0^\infty \exp(-2x)dx$$

2. Determine the convergence of the improper integral

$$\int_{-\infty}^\infty \frac{|x|}{x^2+1}dx$$

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. Let r be a real number, $b > 0$. Show that $\int_b^\infty \frac{1}{x^r} dx$ is convergent if $r > 1$ and divergent if $r \leq 1$.
2. Is the improper integral $\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx$ convergent or divergent?
3. Is the improper integral $\int_{-\infty}^\infty |x| \exp(-x^2) dx$ convergent or divergent?