Middlebury College

Calculus II: Fall 2017

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November 3 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 8.2.
- *Calculus II*, Marsden, Weinstein: Chapter 9.

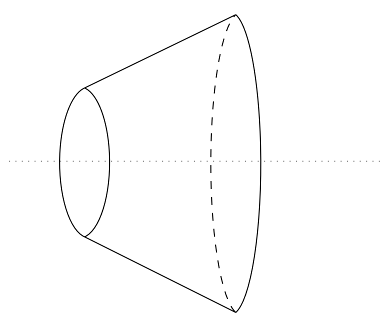
Applications of integration III: Surface Area & Surfaces of Revolution

In this lecture we will investigate further applications of integration. We introduce the Slice Method and determine the surface area and volume of a class of solids known as *surfaces of revolution*.

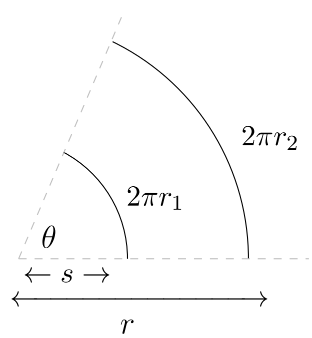
1 Surface Area Suppose that R is a nice surface lying along side the interval $[a, b]$ of the x-axis. We would like to develop a method to compute the surface area of R.

Mathematical workout - flex those muscles

Let F be the circular frustrum obtained by rotating the line segment $y = x$, $1 \leq x \leq 2$, around the x-axis.



Let's determine the surface area of F. If we cut F along a line and unroll it in the plane we obtain a circular sector having radius r and angle θ with a concentric sector of radius $r - s$ removed.



1. What are the numerical values of r_1 and r_2 in the above figure?

2. What are the numerical values of r, s? (*Hint: r, s can be realised as the distance between two points in the plane; which two points?*)

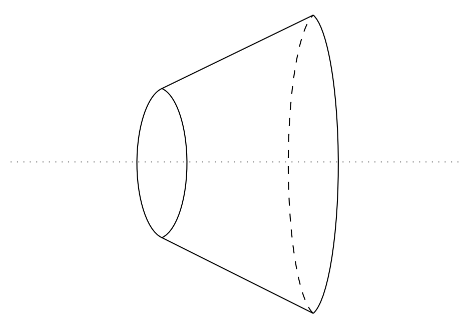
3. Recall that, by the definition of radians, $\theta r = 2\pi r_2$ (similarly, $\theta s = 2\pi r_1$). Hence,

$$\theta = \underline{\hspace{3cm}}.$$

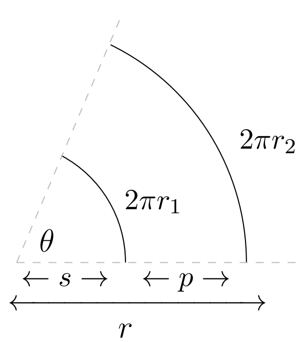
4. Given that the area of a circular sector having angle t and radius R is $\frac{t}{2\pi}\pi R^2$, show that the area of the frustrum F is $3\sqrt{2}\pi$.

Area of a general frustrum Let's generalise our investigation above to the general case: let $y = mx + c$ be a line segment lying in the upper half plane defined between $x = a$ and $x = b$. Assume $m \neq 0$. If we rotate this line segment around the x-axis we obtain a circular frustrum F.

$$y = mx + c$$
$$a \leq x \leq b$$



To determine the surface area of F we proceed as above and determine the area of the ring sector below.



Define $p = r - s$. Then,

$$r_1 = \underline{\hspace{3cm}} \qquad r_2 = \underline{\hspace{3cm}}$$

Since $\theta r = 2\pi r_2$ and $\theta s = 2\pi r_1$, we have

$$\theta p = 2\pi(r_2 - r_1) \quad \Longrightarrow \quad \theta = \underline{\hspace{3cm}}$$

Recall that the length of a straight line segment $y = mx + c$, between $x = x_1$ and $x = x_2$, is $(x_2 - x_1)\sqrt{1 + m^2}$. Hence,

$$s = \underline{\hspace{3cm}} \qquad r = \underline{\hspace{3cm}} \qquad p = \underline{\hspace{3cm}}$$

Finally, the area A of the frustrum F is

$$A = \frac{\theta}{2\pi}\pi(r^2 - s^2) = \frac{\theta}{2}p(r + s) = \underline{\hspace{5cm}}$$

CHECK YOUR UNDERSTANDING

Let $f(x) = mx + c$. Show that the area A of the frustrum F satisfies

$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx$$

Remark 1.1. The integral formula just given for the area of a circular frustrum is also valid when $m = 0$ i.e. when F is a cylinder having radius c and height $(b - a)$: recall that the area of a cylinder having radius c and height h is $2\pi c^2 h$

2 Surface area of surfaces of revolution In this paragraph we will determine the surface area of the class of surfaces known as *surfaces of revolution*.

Definition 2.1. Let $f(x)$ be a nonegative function whose derivative is continuous on the interval $[a, b]$ i.e. $f(x) \geq 0$, for all $a \leq x \leq b$. The surface obtained by rotating the graph $y = f(x)$ about the x-axis is known as a **surface of revolution**.

Remark 2.2. If $g(y)$ is a function, defined on the interval $c \leq y \leq d$, then the surface obtained by revolving the graph $x = g(y)$ about the y-axis is also known as a surface of revolution. The two definitions are equivalent: one is obtained from the other by reflecting acros the line/plane $y = x$.

Example 2.3.

1. Let $f(x) = mx + c$, and $[a, b]$ be an interval on which $f(x)$ is nonegative. Then, the graph of $f(x)$ is a line segment. The surface of revolution about the x-axis is either:

   (a) a circular frustrum (similar to the diagram on p.1 above);

   (b) a circular cone;

   (c) a cylinder.

   Picture

2. Let $r > 0$. Consider the function $f(x) = \sqrt{r^2 - x^2}$. Then, the surface of revolution obtained from $f(x)$ is the sphere having radius r.

   Picture

We can now approximate the surface area of a surface of revolution using a collection of circular frustrums.

Picture

Let S be a surface of revolution obtained from $f(x)$, $a \leq x \leq b$. Choose a natural number n.

1. Subdivide $[a, b]$ into n subintervals having equal length so that the endpoints of each subinterval are

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

2. Define the piecewise linear function $g_n(x)$ as follows:

$$g_n(x) = m_i(x - x_i) + f(x_i), \qquad \text{when } x_{i-1} \leq x \leq x_i.$$

   Here

$$m_i = \underline{\qquad\qquad\qquad}$$

   The piecewise linear function $g_n(x)$ provides an approximation to the graph of $f(x)$

3. Then, the surface of revolution S is approximated by a collection of n circular frustrums $F_1, \ldots, F_n$ as in the above diagram. Moreover,

$$\text{Surface area of } F_i = \underline{\hspace{6cm}}$$

4. Hence, the surface area of S is obtained as the limit

$$\text{Surface area of } S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \int_{x_{i-1}}^{x_i} \sqrt{1+(g_n'(x))^2} g_n(x)dx$$

$$= \underline{\hspace{7cm}}$$

Example 2.4. 1. The surface area A of the surface of revolution about the x-axis obtained from $f(x) = 2\sqrt{x}$ when $1 \leq x \leq 2$ is

$$A = 2\pi \int_1^2 \sqrt{1+\frac{1}{x}} 2\sqrt{x}dx = 4\pi \int_1^2 \sqrt{x+1}dx = 4\pi \left[\frac{2}{3}(x+1)^{3/2}\right]_1^2 = \frac{8\pi}{3}\left(\sqrt{27}-\sqrt{8}\right)$$

2. Let $f(x) = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$. Then, the surface of revolution obtained from $f(x)$ is the sphere having radius r. Let's compute its surface area A. First we compute

$$f'(x) = \underline{\hspace{4cm}}$$

Then, we have

$$A = 2\pi \int_{-r}^{r} \sqrt{1+(f'(x))^2} f(x)dx$$

$$= \underline{\hspace{7cm}}$$

$$= \underline{\hspace{7cm}}$$

$$= \underline{\hspace{7cm}}$$

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problem. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. Determine the surface area of the surface of revolution about the x-axis obtained from the graph $y = \sqrt{x+1}$ when $0 \leq x \leq 2$. Draw sketch of this surface.
2. Determine the surface area of the surface of revolution about the x-axis obtained from the graph $y = \cos(x)$ when $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Draw a sketch of this surface.
3. Determine the surface area of the surface of revolution about the y-axis (*careful!*) obtained from the graph of $f(x) = x^{1/3}$ when $1 \leq x \leq 3$. Draw a sketch of this surface.
4. Challenge! Use the slice method to determine a formula for the volume of the surface of revolution about the x-axis obtained from $f(x)$, when $a \leq x \leq b$.