Calculus II: Fall 2017

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November 30 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Power Series*, Integral Calculus, Khan Academy

Power Series

In this lecture we will investigate how to use series to solve differential equations. This will lead to the notion of a *power series representation of a function*.

1 An impossible integral? Recall that the antiderivative problem

$$\int \exp(-x^2)dx$$

does not admit an elementary function solution. However, the Fundamental Theorem of Calculus states that the integral function

$$f(x) = \int_0^x \exp(-s^2)ds$$

is an antiderivative of $\exp(-x^2)$, which means

$$\frac{d}{dx}f(x) = \exp(-x^2).$$

This leads to a basic question:

> **Problem:** How can we *represent* the function $f(x)$ in a way that allows us to understand its properties more clearly (i.e. not as an integral function!)?

Recall that

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\implies \quad \exp(-x^2) = 1 + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{n!} = \rule{6cm}{0.4pt} \qquad (*)$$

A reasonable guess, therefore, may be to represent $f(x)$ as an infinite series, in a similar way to how we defined $\exp(x)$. We give such expressions a special name.

Definition 1.1. A **power series** is a series of the form

$$\sum_{n \geq 0} c_n(x-c)^n$$

where $c_0, c_1, c_2, \dots$ and c are constant, and x is a variable. We call c the **centre** of the power series.

Remark 1.2. Observe that a power series is completely determined by its centre c and the coefficients $c_0, c_1, c_2, c_3, \ldots$: any two power series possessing the same centre and coefficients are the same power series.

Mathematical workout - flex those muscles!

Let's $f(x)$ be an antiderivative of $\exp(-x^2)$ satisfying $f(0) = 1$. We are going to try to represent $f(x)$ as a power series centred at 0,

$$f(x) = \sum_{n \geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

1. Use the condition $f(0) = 1$ to determine a_0.

2. Let's assume that we can differentiate the power series term-by-term, so that

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

Use the power series expansion of $\exp(-x^2)$ above to determine the following coefficients:

$a_1 =$ ________ $a_2 =$ ________ $a_3 =$ ________ $a_6 =$ ________ $a_7 =$ ________

3. Spot the pattern! Write down the general expression for a_n:

$$a_n = \begin{cases} \underline{\hspace{4cm}}, & n \text{ even}, \\ \underline{\hspace{4cm}}, & n = 2k+1 \text{ odd}. \end{cases}$$

4. Use the previous calculations to complete the following power series representation for $f(x)$, the antiderivative of $\exp(-x^2)$ satisfying $f(0) = 1$:

$$f(x) = \underline{\hspace{1.5cm}} + \sum_{k=0}^{\infty} \underline{\hspace{4cm}} x^{2k+1}$$

2 Convergence of power series The power series introduced above is a candidate for an antiderivative of $\exp(-x^2)$. However, there are some issues we must address:

1. for which x is the power series $f(x)$ a *well-defined function*? We must consider this problem because $f(x)$ is defined using a series, and we need to check for which x is the series convergent.
2. is it true that we can differentiate a power series *term-by-term*? If so, then we can be sure that the power series representation of $f(x)$ is a valid solution to the antiderivative problem $\int \exp(-x^2)dx$ (wherever it is defined).

We will now focus on the first problem above. Recall the **Ratio Test**:

> Let $\sum b_n$ be a series such that $b_n \neq 0$, for every n. Let $L = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|$. Then,
>
> - $\sum b_n$ converges if $L < 1$,
> - $\sum b_n$ diverges if $L > 1$,
> - if $L = 1$ then no conclusion can be made and further investigation is required.

You've shown above that

$$f(x) = \_\_\_\_ + \sum_{k=0}^{\infty} \_\_\_\_\_\_\_\_\_\_\_\_ x^{2k+1} \qquad (**)$$

Letting $y = x^2$, we can rewrite the power series

$$\_\_\_\_ + x\left(\sum_{k=0}^{\infty} \_\_\_\_\_\_\_\_\_\_\_\_ y^k\right)$$

In particular, **this series converges if and only if the series**

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} y^k \qquad (***)$$

converges. Using the Ratio Test, for fixed $y \neq 0$, we determine

$$\left|\frac{(-1)^{k+1}y^{k+1}}{(k+1)!(2k+3)} \frac{k!(2k+1)}{(-1)^k y^k}\right| = \frac{|y|(2k+1)}{(k+1)(2k+3)} \to 0 \text{ as } k \to \infty$$

Of course, the series $(***)$ is convergent when $y = 0$. Hence, for any y the series $(***)$ is convergent, which implies that the series $(**)$ is convergent, for any x.

> The function
>
> $$f(x) = \_\_\_\_ + \sum_{k=0}^{\infty} \_\_\_\_\_\_\_\_\_\_\_\_ x^{2k+1}$$
>
> is defined for every x.

Remark 2.1. This is not so surprising due to how we defined $f(x)$ in the first place: as an antiderivative of $\exp(-x^2)$

CHECK YOUR UNDERSTANDING

Suppose you are given the power series

$$\sum_{n=0}^{\infty} c_n (x-c)^n$$

1. Let $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$. By considering the Ratio Test applied to the power series, explain why the power series
   (a) converges if $|x-c| < R$,
   (b) diverges if $|x-c| > R$,
   (c) further investigation is required if $|x-c| = R$.

2. Complete the following statement:

   Let $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$. Then, the power series $\sum_{n=1}^{\infty} c_n(x-c)^n$ is **convergent** if

   $$\_\_\_\_\_\_\_\_ < x < \_\_\_\_\_\_\_\_$$

This investigation leads us to the following important definition.

Definition 2.2. Let $\sum_{n=0}^{\infty} c_n(x-c)^n$ be a power series. Define the **radius of convergence** to be

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$$

Here, R is either a nonegative real number or equal to $+\infty$. In this latter case we say that the **radius of convergence is infinite**.

We have the following immediate consequence:

Let $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$ be the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n(x-c)^n$. Then, the power series

- converges if $|x-c| < R$, (i.e. $c - R < x < c + R$)
- diverges if $|x-c| > R$, (i.e. $x < c - R$ or $x > c + R$)
- if $|x-c| = R$ then further investigation is required.

Example 2.3. 1. Consider the exponential series

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

This is a power series centred at $c = 0$, and $c_n = \frac{1}{n!}$, for $n = 0, 1, 2, 3, 4, \ldots$. The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} (n+1) = +\infty$$

Hence, we recover the fact already established that $\exp(x)$ is well-defined for all x.

2. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

This is a power series centred at $c = 1$ and $c_n = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

Hence, the power series

(a) converges when ______________ i.e. when ______________, and

(b) diverges when ______________ i.e. when ____________________.

If ______________ then __________ or __________ and we have two separate cases to consider for convergence.

- __________: In this case the power series is

____

This is ______________ by ____________________.

- __________: In this case the power series is

____

This series is ______________ by ____________________

Hence, the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ is convergent when ______________, and divergent otherwise.

3. Consider the power series

$$\sum_{n=0}^{\infty} n!(1-x)^n = \sum_{n=0}^{\infty} n!(-1)^n(x-1)^n$$

We have coefficients $c_n = (-1)^n n!$. The centre of the power series is $c = 1$ and the radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n n!}{(-1)^{n+1}(n+1)!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence, the radius of convergence is $R = 0$. Thus, the series converges at $x = 1$ and diverges for $x \neq 1$.

Check your understanding

Consider the power series

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n(n+1)}$$

Determine

(a) the centre c,

(b) the radius of convergence R,

(c) the largest interval on which the power series converges.

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

Determine the centre c, radius of convergence R, and the largest interval on which the power series converges.

1. $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n^2+1}$
2. $\sum_{n=0}^{\infty} \frac{(-x)^{2n}}{(2n)!}$
3. $\sum_{n=0}^{\infty} \frac{(4-2x)^n}{2n+1}$