Calculus II: Fall 2017

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November 2 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 8.2.
- *Calculus II*, Marsden, Weinstein: Chapter 9.
- *Integral Calculus*, Khan Academy: Volumes using calculus.

Applications of integration II: The Slice Method

In this lecture we will see several applications of integration. We introduce the Slice Method and determine the surface area and volume of a class of solids known as *surfaces of revolution*.

1 Definite integration, contd. First, we will complete our discussion of how the techniques of integration we have developed affect definite integration problems.

Method of integration by parts

Let's see how definite integration is affected by the method of integration by parts.

Check your understanding

Let $f(x) = x$ and $g(x) = \log(x)$.

1. Explain why
$$\int_1^2 (\log(x) + 1)\, dx = f(2)g(2) - f(1)g(1)$$
(*Hint: consider the right hand side as an antiderivative of a function*)

2. Use the previous problem to complete the following statement:

Let $f(x)$, $g(x)$ be differentiable functions. Then, $$\int_a^b f(x)g'(x)dx = \underline{\hspace{4cm}} - \int_{\_\_}^{\_\_} \underline{\hspace{3cm}}\, dx$$

Example 1.1. 1. Recall the formula for the arc length L of $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$:

$$L = \int_{-2}^{2} \sqrt{1+4x^2}dx = 2\int_{0}^{2} \sqrt{1+4x^2}dx$$

Let $x = \frac{1}{2}\tan(t)$, $\frac{dx}{dt} = \frac{1}{2}\sec^2(t)$. Then, when

$$x = 0, \quad t = \underline{\hspace{4cm}}$$

$$x = 2, \quad t = \underline{\hspace{4cm}}$$

As

$$\sqrt{1+4x^2}\frac{dx}{dt} = \underline{\hspace{6cm}}$$

the method of inverse trig. substitution gives

$$L = \int_{t=0}^{t=\arctan(4)} \sec^3(t)dt$$

Using integration by parts, and the fact that

$$\frac{d}{dt}\log(\sec(t) + \tan(t)) = \sec(t),$$

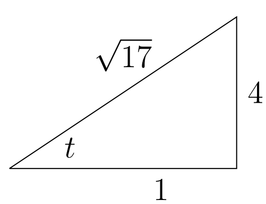
it can be shown (try it yourself!) that an antiderivative of $\sec^3(t)$ is

$$\frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\log|\sec(t) + \tan(t)|$$

Hence,

$$L = \int_{t=0}^{t=\arctan(4)} \sec^3(t)dt = \left[\frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\log|\sec(t)+\tan(t)|\right]_{t=0}^{t=\arctan(4)}$$

If $4 = \tan(t)$ then we can use the triangle



to compute

$$L = 2\sqrt{17} + \frac{1}{2}\log(\sqrt{17}+4)$$

2. Using integration by parts we can determine

$$\int_{0}^{\pi/2} \cos^3(x)dx$$

as follows: let

$$\begin{array}{ll} f = \cos^2(x) & g' = \cos(x) \\ f' = -2\sin(x)\cos(x) & g = \sin(x) \end{array}$$

so that, by the method of integration by parts,

$$\int_0^{\pi/2} \cos^3(x)dx = \left[\cos^2(x)\sin(x)\right]_0^{\pi/2} + 2\int_0^{\pi/2} \sin^2(x)\cos(x)dx = 2\int_0^{\pi/2} \sin^2(x)\cos(x)dx$$

since $\sin(0) = \cos(\pi/2) = 0$. Let $u = \sin(x)$, $\frac{du}{dx} = \cos(x)$. Then,

$$x = 0 \implies u = 0$$
$$x = \pi/2 \implies u = 1$$

Since

$$\sin^2(x)\cos(x) = u^2\frac{du}{dx}$$

the method of substitution gives

$$2\int_0^{\pi/2} \sin^2(x)\cos(x)dx = \int_0^1 2u^2 du$$

Hence,

$$\int_0^{\pi/2} \cos^3(x)dx = \frac{2}{3}$$

2 The Slice Method & Volume

We turn our attention back to applications of integration. In this paragraph we will see how integration can be used to determine the volume of a class of solids. Our approach is based on the following method:

The Slice Method

Let S be a nice solid region lying alongside some interval $[a, b]$ of the x-axis. For any real number c, let P_c be the plane perpendicular to the x-axis. Suppose that

1. S lies between the planes P_a and P_b,

2. for each $a \leq x \leq b$, the area of the slice of S cut out by P_x is $A(x)$.

Then,

$$\text{Volume of } S = \int_a^b A(x)dx$$

Idea of Slice Method

Example 2.1. Consider the solid ball B having radius $r > 0$. Assume that B lies between the planes P_{-r} and P_r and that the point $x = 0$ is the center of B. Then, for each $-r \leq x \leq r$, the area of the slice of B cut out by P_x is

$$\begin{aligned} A(x) &= \text{area of circle having radius } \sqrt{r^2 - |x|} \\ &= \pi(r^2 - x^2) \end{aligned}$$

Hence,

$$\text{Volume of } B \;= \int_{-r}^{r} \pi(r^2 - x^2)dx = \frac{4\pi}{3}r^3$$

Check your understanding

We will compute the volume of the solid region S lying between a circular cone and a cylinder.

1. Let C_1 be the circular cone obtained by rotating the line $y = x$, $0 \leq x \leq 2$, around the x-axis. Determine the cross-sectional area of the slice of C_1 cut out by P_x, where $x \geq 0$.

2. Let C_2 be the cylinder of radius 2 parallel to, and centered along, the x-axis, where $0 \leq x \leq 2$. Determine the cross-sectional area of the slice of C_2 cut out by P_x, where $x \geq 0$.

3. Determine the volume V of the solid region S lying between C_1 and C_2.

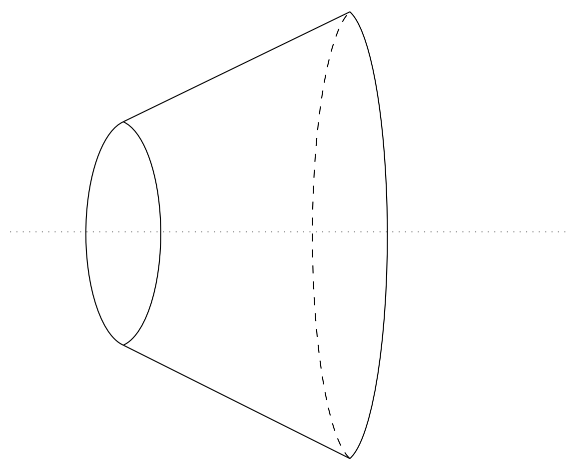
4. For each $0 \leq x \leq 2$, determine the cross-sectional area $A(x)$ of the slice of S cut out by P_x, $0 \leq x \leq 2$.

5. Verify the Slice Method i.e. confirm that $\int_0^2 A(x)dx = V$

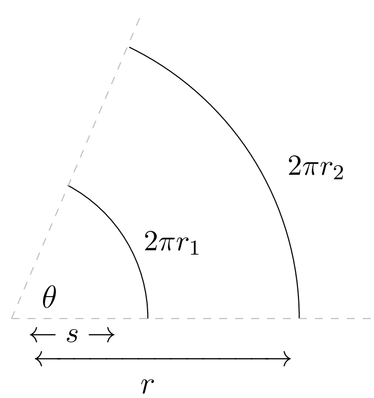
3 Surface Area Suppose that R is a nice surface lying along side the interval $[a, b]$ of the x-axis. We would like to develop a method to compute the surface area of R.

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES

Let F be the circular frustrum obtained by rotating the line segment $y = x$, $1 \le x \le 2$, around the x-axis.



Let's determine the surface area of F. If we cut F along a line and unroll it in the plane we obtain a circular sector having radius r and angle θ with a concentric sector of radius $r - s$ removed.



1. What are the numerical values of r_1 and r_2 in the above figure?

2. Recall that, by the definition of radians, $\theta r = 2\pi r_2$. Complete the following statement

$$\theta\underline{\hspace{2cm}} = 2\pi\underline{\hspace{1cm}}$$

3. Use the previous problem to write θ in terms of s.

4. Deduce an expression for r in terms of s.

5. Given that the area of a circular sector having angle t and radius R is $\frac{t}{2\pi}\pi R^2$, show that the area of the frustrum F is $\pi s(r_1 + r_2)$. (*Hint: realise the area of the frustrum F as the difference of two circular sectors*).