Middlebury College

Calculus II: Fall 2017

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November 29 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.3-9.6.
- *Calculus II*, Marsden, Weinstein: Chapter 8.

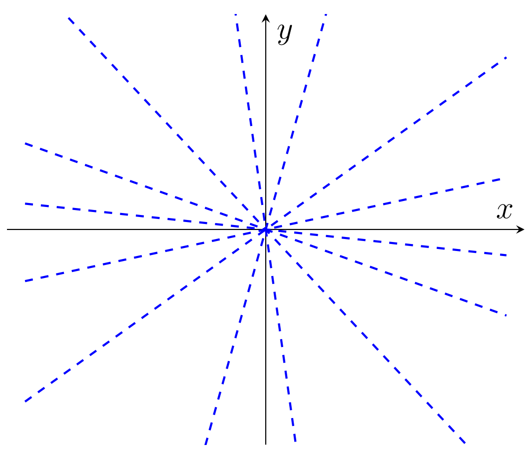
Differential Equations VI

In this lecture we will investigate numerical methods to solve first-order differential equations. We will describe *Euler's method* for numerically solving first-order differential equations by first considering the *orthogonal trajectory problem*.

1 Orthogonal trajectories Consider the family of straight lines

$$y = kx, \quad k \text{ constant.}$$

By a *family*, we mean the collection of all straight lines obtained as we vary the parameter k.



The **Orthogonal Trajectory Problem** is defined as follows:

Problem: is it possible to find a curve C, or a family of curves C_j (depending on a parameter j), such that each curve C_j is orthogonal (i.e. perpendicular) to *every* straight line $y = kx$?

Check your understanding

1. Which of the following curves are orthogonal (i.e. perpendicular) to *every* straight line $y = kx$?

$$(A): \quad y = x^2, \quad (B): \quad x^2 + y^2 = 1, \quad (C): \quad xy = 1, \quad (D): \quad \frac{x^2}{5} + \frac{y^2}{5} = 1, \quad (E): \quad y = x$$

2. Give equations defining three further curves that are orthogonal to the family of straight lines $y = kx$.

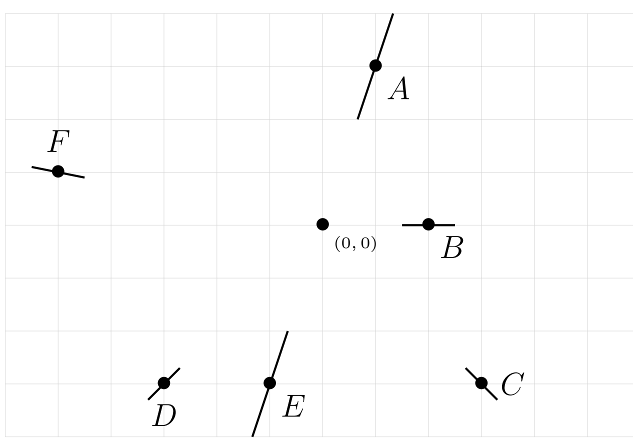
3. Complete the statement:

> ($*$) The family of ______________ centred at __________ is orthogonal to every straight line $y = kx$.

Let's think a bit more closely about what it means for two curves to be *orthogonal*.

Mathematical workout - flex those muscles!

1. Each of the points A, B, C, D, E, F lies on a line $y = kx$. What is the slope of the line perpendicular to the line segment drawn at each point?



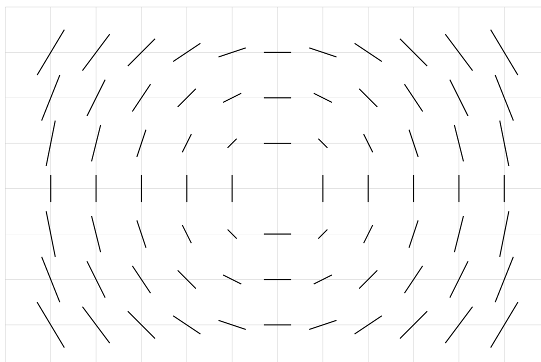
Point	Perp. Slope
A	
B	
C	
D	
E	
F	

2. What is the general expression for the slope of the line perpendicular to the straight line $y = kx$ passing through the points (x, y)?

3. Write down a differential equation that describes a curve C orthogonal to *every* line $y = kx$. (*Hint: the tangent line to the curve C should be perpendicular to $y = kx$ at any point (x, y) on C*)

4. Solve the differential equation given above. How does your solution relate to your guess for the statement $(*)$ above?

Remark 1.1. The differential equation $\frac{dy}{dx} =$ __________ that you gave above can be used to generate a **phase diagram**: at each point (x, y) in the plane, draw a short line segment having slope __________, which is prescribed by the differential equation we are looking to solve. In this way you obtain the following diagram



Qualitatively, the phase diagram *shows* us the solutions to the differential equation: if we want the solution passing through $P = (a, b)$ then we imagine that we have placed a tiny ball bearing at P and have it move an infinitesimal distance *anti-clockwise* in the direction prescribed by the line segment placed at P. Then, the ball bearing has moved to a new point P' and we have it move an infinitesimal distance anti-clockwise in the direction prescribed by the line segment placed at P'. Poceeding in the fashion we *see* that the solution to the differential equation passing through P should be described by a circular curve, in agreement with what we found using separation of variables.

Definition 1.2. Given a differential equation of the form

$$\frac{dy}{dx} = F(x, y),$$

where $F(x, y)$ is an expression involving (possibly) both x and y, the associated **phase diagram** obtained as follows: at each point (x, y) in the plane, draw a short line segment having slope $F(x, y)$.

2 Euler's Method (Non-examinable)

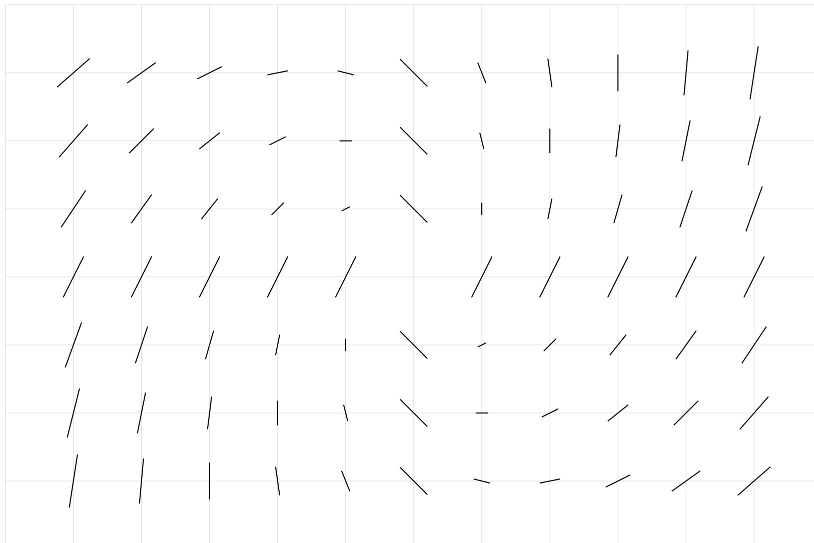
The above discussion suggests a numerical approach to approximating solutions to a given differential equation. We will highlight this approach with an example.

Suppose that we have the differential equation

$$\frac{dy}{dx} = \frac{2x + y}{x - y}$$

This is a first-order differential equation which is neither separable nor linear. The associated phase diagram is drawn below: qualitatively the solutions look like they will be *periodic* and described by curves shaped like ellipses.

In an advanced differential equations course, or once you've learned some methods of linear algebra (e..g eigenvalues and diagonalisation), you can verify this qualitative guess.



Euler's method allows us to give a numerical approach to solving this differential equation by replacing the actual solution curve by a polygonal line that follows the phase diagram. Suppose we wish to approximate the solution to

$$\frac{dy}{dx} = \frac{2x+y}{x-y}, \quad y(0) = -2,$$

between $x = 0$ and $x = 4$. Let's subdivide the interval $[0, 4]$ into $n = 10$ (this is an arbitrary choice) intervals and define

$$x_0 = 0, \quad x_i = x_0 + \frac{4i}{10} = \frac{4i}{10}, \quad i = 1, \dots, 10$$

Now, define

$$\begin{aligned}
y_0 &= -2 \\
y_1 &= \tfrac{dy}{dx}(x_0, y_0)\tfrac{4}{10} + y_0 \\
y_2 &= \tfrac{dy}{dx}(x_1, y_1)\tfrac{4}{10} + y_1 \\
y_3 &= \tfrac{dy}{dx}(x_2, y_2)\tfrac{4}{10} + y_2 \\
&\vdots \\
y_{10} &= \tfrac{dy}{dx}(x_9, y_9)\tfrac{4}{10} + y_9
\end{aligned}$$

Observe that

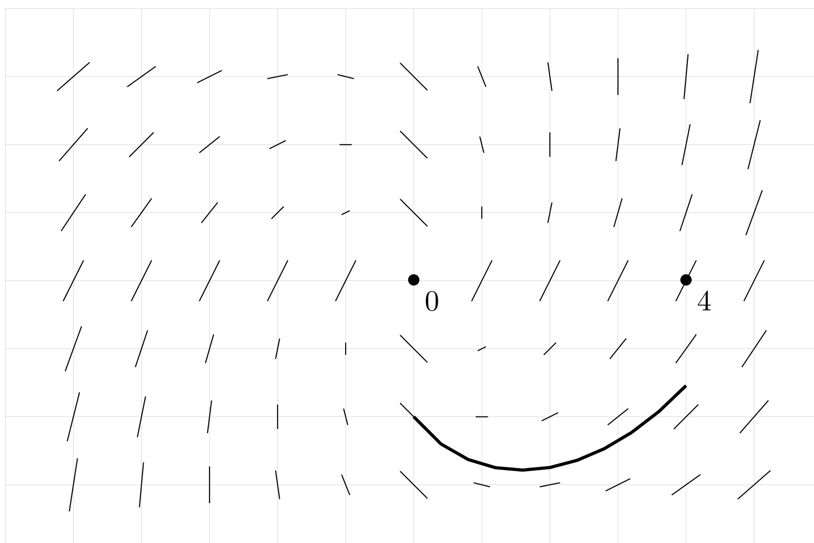
$$y_i - y_{i-1} = \left(\frac{dy}{dx}(x_{i-1}, y_{i-1})\right)(x_i - x_{i-1})$$

In particular, if we plot the points $(x_0, y_0), (x_1, y_1), \dots, (x_{10}, y_{10})$ and join them using line segments, then the slope of the i^{th} line segment has slope equal to the slope of the line segment at (x_{i-1}, y_{i-1}) in the phase diagram. The algorithm described above is called the **(10-step) Euler method**.

Let's see a convenient way to tabulate this information:

	x_i	y_i	$x_{i+1} = x_i + \frac{4}{10}$	$y_{i+1} = y_i + \left(\frac{2x_{i-1}+y_{i-1}}{x_{i-1}-y_{i-1}}\right)\frac{4}{10}$
$i = 0$	0	-2	0.4	-2.4
$i = 1$	0.4	-2.4	0.8	-2.6286
$i = 2$	0.8	-2.6286	1.2	-2.7486
$i = 3$	1.2	-2.7486	1.6	-2.7839
$i = 4$	1.6	-2.7839	2	-2.7459
$i = 5$	2	-2.7459	2.4	-2.6402
$i = 6$	2.4	-2.6402	2.8	-2.4688
$i = 7$	2.8	-2.4688	3.2	-2.2311
$i = 8$	3.2	-2.2311	3.6	-1.9241
$i = 9$	3.6	-1.9241	4	-1.5420
$i = 10$	4	-1.5420		

Now, we plot these points on the phase diagram and join them using line segments.



Remark 2.1. 1. In practice, Euler's method requires a substantial number of computations to provide effective approximate solutions to a first-order differential equation. More effective numerical methods are provided by *Runge-Kutta methods*.

2. Euler's method appears in the recent movie *Hidden Figures*: it is used to help bring astronaut John Glenn back to Earth from ortbit.

Mathematical workout - flex those muscles

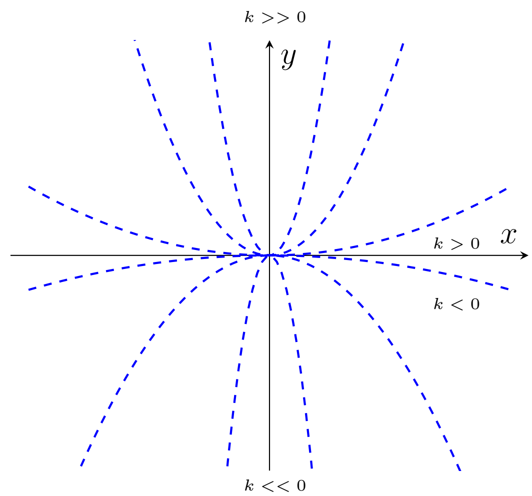
Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

Problem:

Consider the family of parabolic curves

$$y = kx^2, \quad k \text{ constant.}$$

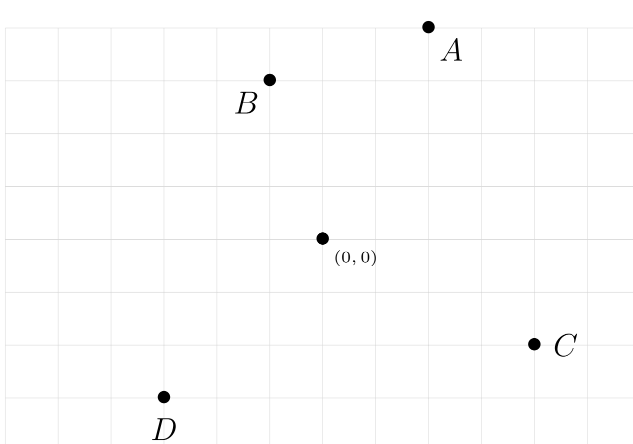
By a family, we mean the collection of all parabolas obtained by varying k.



1. Determine k so that the parabola $y = kx^2$ passes through the points $A = (2, 4)$, $B = (-1, 3)$, $C = (4, -2)$, $D = (-3, -3)$.

	A	B	C	D
k				

2. Draw the tangent line to the curve $y = kx^2$ passing through the points A, B, C, D. Draw a line segment perpendicular to each tangent line and passing through each point, and mark down the slope of the line perpendicular to the line segment.



Point	Perp. Slope
A	
B	
C	
D	

3. Determine a general expression for the slope of the perpendicular line segments drawn above.

4. Write down a differential equation that describes a curve C orthogonal to *every* line $y = kx^2$. (*Hint: the tangent line to the curve C should be perpendicular to the tangent line to $y = kx^2$ at any point (x, y) on C*)

5. Solve the differential equation given above. What does your solution tell you?