Middlebury College

Calculus II: Fall 2017

Contact: `gmelvin@middlebury.edu`

November 27 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.3-9.6.
- *Calculus II*, Marsden, Weinstein: Chapter 8.

Differential Equations V

In this lecture we will see several examples of differential equations arising in nature. We will see that first order differential equations can be used to model

(i) chemical reactions and mixing problems,

(ii) the shape of a hanging cable,

(iii) the current in an electric circuit,

(iv) predator-prey systems.

Remark 0.1. The following material is **non-examinable**. It is intended to highlight an array of interesting situations in which first-order differential equations appear. It will not appear on an exam.

1 Chemical reaction rates In this paragraph we will see how first-order differential equations appear in models describing chemical kinetics.

Let B and C be two substances that combine to produce a new chemical A:

$$B + C \to A \tag{$*$}$$

The **Law of Mass Action** states:

"the instantaneous rate at which A is formed is proportional to the product of the concentrations of B and C at time t"

Let $a(t)$, $b(t)$, $c(t)$ be, respectively, the concentrations of A, B and C at time t. Then, the rate of reaction is described by the differential equation

$$\frac{da}{dt} = kbc$$

In chemistry, the equation $(*)$ signifies that a single molecule of B and a single molecule of C produce a single molecule of A. If we assume that chemical A is not initially present then we obtain

$$\frac{da}{dt} = k(b_0 - a)(c_0 - a) \tag{1.1}$$

Here $b_0 = b(0)$ and $c_0 = c(0)$ are the initial concentrations of B and C, respectively.

(1.1) is a *separable equation*: we rearrange to get

$$\frac{1}{(b_0-a)(c_0-a)}\frac{da}{dt}=k$$

Hence,

$$\int \frac{1}{(b_0-a)(c_0-a)}da=\int kdt$$

If $b_0=c_0$ then

$$\frac{1}{b_0-a}=kt+C \qquad \Longrightarrow \qquad a(t)=b_0-\frac{b_0}{kb_0t+1}.$$

Here we use the assumption $a(0)=0$ to determine the constant C.

If $b_0\neq c_0$ then, using partial fractions, one finds

$$\frac{1}{c_0-b_0}\left(\ln(b_0-a)-\ln(c_0-a)\right)=kt+C, \quad \text{with } C \text{ constant.}$$

Rearranging, and using $a(0)=0$, gives

$$\boxed{a(t)=b_0c_0\left(\frac{e^{k(c_0-b_0)t}-1}{c_0e^{k(c_0-b_0)t}-b_0}\right)}$$

Check your understanding

Complete the following statements and provide a physical justification of your answer:

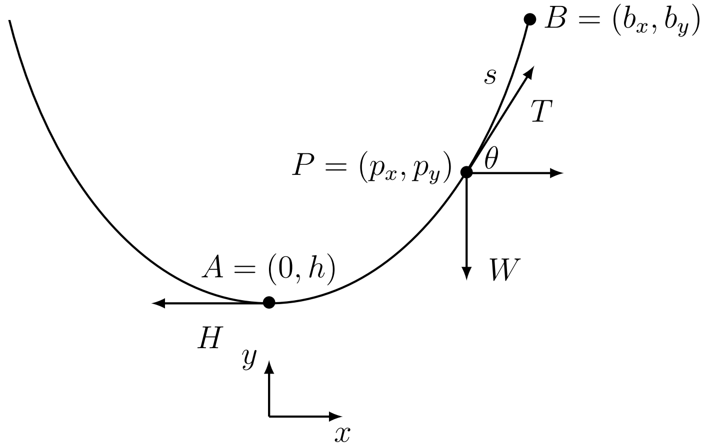
$(c_0>b_0)$: $a(t)\to$ __________ as $t\to\infty$.

$(b_0>c_0)$: $a(t)\to$ __________ as $t\to\infty$.

2 Suspended cables In this paragraph we will introduce a differential equation describing a freely hanging cable having constant density δ grams per cm of cable.

Assume that the cable is sufficiently flexible so that it does not experience any bending force and that the cable is at a stable equilibirum (i.e. it isn't moving, touching the ground etc.).

The situation is modelled below: our aim is to describe the curve $y=f(x)$ modelling the hanging cable.



There are three forces acting on the cable at a point P:

(i) H = horizontal tension pulling at A

(ii) T = tangential tension pulling at P

(iii) $W = \delta s$ = weight of s cm of cable (measured from B) having density δ

Assume that the lowest point of the cable is h cm off the ground. To be in equilibrium the horizontal and vertical components of T must balance H and W respectively:

$$T\cos\theta = H, \qquad T\sin\theta = W = \delta s$$

Hence,

$$\frac{dy}{dx} = \text{slope of tangent line to curve} = \tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{\delta s}{H}$$

Using the arc length formula, we have

$$s = \int_{b_x}^{p_x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Hence,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\delta}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{2.1}$$

Let

$$w = \frac{dy}{dx}$$

Then, (2.1) becomes

$$\frac{dw}{dx} = \frac{\delta}{H}\sqrt{1 + w^2}$$

This is a *separable equation*: we rearrange to get

$$\int \frac{1}{\sqrt{1 + w^2}} dw = \frac{\delta}{H}\int dx$$

Using the inverse trigonometric substitution $w = \tan(t)$ we obtain

$$\sqrt{1 + w^2} + w = A\exp(\delta x/H) \tag{2.2}$$

Here's a **!!NIFTY TRICK!!**: multiplying both sides by $1 = \frac{\sqrt{1+w^2}-w}{\sqrt{1+w^2}-w}$, you can show that

$$\sqrt{1+w^2} - w = A\exp(-\delta x/H) \tag{2.3}$$

Thus, combining (2.2) and (2.3) we find

$$w = \frac{A}{2}\left(\exp(\delta x/H) - \exp(-\delta x/H)\right) = \frac{A}{2}\left(e^{\delta x/H} - e^{-\delta x/H}\right)$$

Recalling that $w = \frac{dy}{dx}$, we find

$$y = \frac{AH}{2\delta}\left(e^{\delta x/H} + e^{-\delta x/H}\right)$$

Since $(0, h)$ is a point on the curve

$$h = y(0) = \frac{AH}{\delta} \qquad \Longrightarrow \qquad A = \frac{h\delta}{H}$$

Hence, the curve describing a freely hanging cable is

Catenary curve:

$$y = \frac{h}{2}\left(e^{\delta x/H} + e^{-\delta x/H}\right)$$

Remark 2.1. The problem of determining the curve describing a freely hanging cable was posed as a challenge problem by Jakob Bernoulli in the late 17th century. Solutions were obtained by Gottfried Leibniz, Christiaan Huygens and Johann Bernoulli in 1691.

The graph $y = \frac{1}{2}\left(e^x + e^{-x}\right)$ is called a **catenary curve**. The word *catenary* is derived from the Latin word *catena*, which means "chain". The English word "catenary" is usually attributed to Thomas Jefferson, who wrote in a letter to Thomas Paine on the construction of an arch for a bridge:

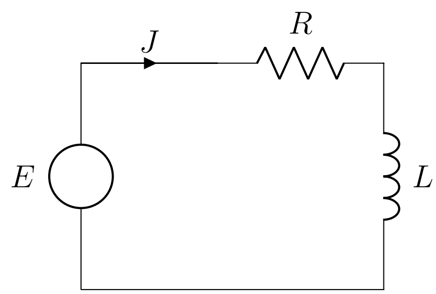
> *I have lately received from Italy a treatise on the equilibrium of arches, by the AbbÃ© Mascheroni. It appears to be a very scientifical work. I have not yet had time to engage in it; but I find that the conclusions of his demonstrations are, that every part of the catenary is in perfect equilibrium.*

(So says Wikipedia)

3 A simple electric circuit

In this paragraph we introduce a differential equation describing the current in a simple electric circuit at time t.

Consider an electric circuit containing an electromotive force (e.g. a battery or a generator) that produces a voltage of $E(t)$ volts and a current of $J(t)$ amperes at time t. The circuit also contains a resistor with a (constant) resistance of R ohms and an inductor with an (constant) inductance of L henries.



Such a circuit is known as a *RL filter* or *RL network*. Ohm's Law states: *the drop in voltage due to the resistor is JR; the drop in voltage due to the inductor is $L\frac{dJ}{dt}$.* Kirchoff's Law states: *the sum of the voltage drops is equal to the supplied voltage $E(t)$.* Hence,

$$L\frac{dJ}{dt} + JR = E(t) \tag{3.1}$$

This is a *linear first-order differential equation*. If $E(t) \equiv E$ is constant (i.e. the battery is producing a constant voltage) then we rearrange to obtain a *separable equation*:

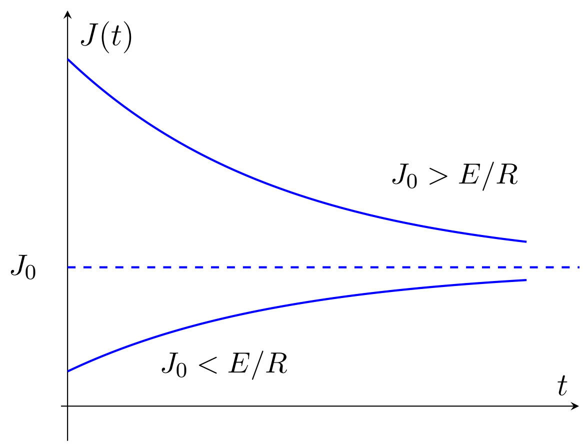
$$\frac{L}{JR - E}\frac{dJ}{dt} = -1$$

Integrating (LHS with respect to J, RHS with respect to t) we find the general solution

$$\frac{L}{R}\log(JR - E) = -t + C \qquad \Longrightarrow \qquad JR - E = A\exp(-tR/L), \quad \text{where } A \text{ is constant.}$$

If the initial current in the circuit is $J(0) = J_0$ then we compute $A = E - RJ_0$. Hence,

$$\boxed{J(t) = \frac{E}{R} + \left(J_0 - \frac{E}{R}\right)\exp(-tR/L)}$$



Suppose $E(t) = E_0\sin(\omega t)$ is a sinusoidal voltage. Here E_0 is the maximum voltage amplitude. For example, such a situation will arise when the voltage source is an alternator - this is a generator with a coil rotating in a magnetic field. All household current, produced from a power utility, takes this form.

Then, (3.1) can be rearranged to

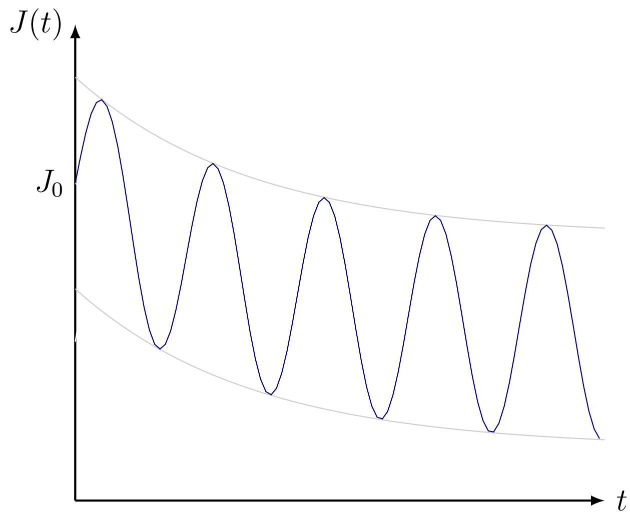
$$\frac{dJ}{dt} + \frac{JR}{L} = \frac{E_0}{L}\sin(\omega t)$$

Check your understanding

Use integrations by parts twice to show that

$$J(t) = \frac{E_0}{L} \frac{1}{(R/L)^2 + \omega^2} \left(\frac{R}{L} \sin(\omega t) - \omega \cos(\omega t) \right) + C \exp(-tR/L)$$

A graph of a particular solution (such that $J_0 > E_0/R$) is plotted below.



4 Predator-prey systems In this paragraph we introduce a differential equation describing a predator-prey system (experiencing negligible external factors). These models are important in ecology for predicting and studying variations in populations.

Let x denote a population of predators and y denote a population of prey upon which x feeds. We assume that $x \equiv x(t)$, $y \equiv y(t)$ vary with time t. Consider the following model (known as *Lotka-Volterra model*):

(i) The prey increases by natural population growth at a rate by (here b is a positive birth rate constant), but decreases at an instantaneous rate proportional to the number of predators and the number of prey $-rxy$ (here r is a positive death rate constant). Thus, the combined instantaneous rate of change of prey is

$$\frac{dy}{dt} = by - rxy \tag{4.1}$$

(ii) The predators's population decreases at a rate $-sx$ proportional to their number due to natural decay (i.e. x is a positive starvation rate constant) and increases at a rate cxy proportional to the number of predators and the number of prey (here c is a positive growth rate constant). Hence,

$$\frac{dx}{dt} = -sx + cxy \qquad (.42)$$

We can consider the population of prey y as dependent upon the population of predators x.

Aim: understand the relationship between y and x.

The Chain Rule implies that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Substituting (4.1), (4.2) shows that

$$\frac{dy}{dx} = \frac{by - rxy}{-sx + cxy} = \frac{b - rx}{x} \cdot \frac{y}{-s + cy}.$$

This is a *separable differential equation.*

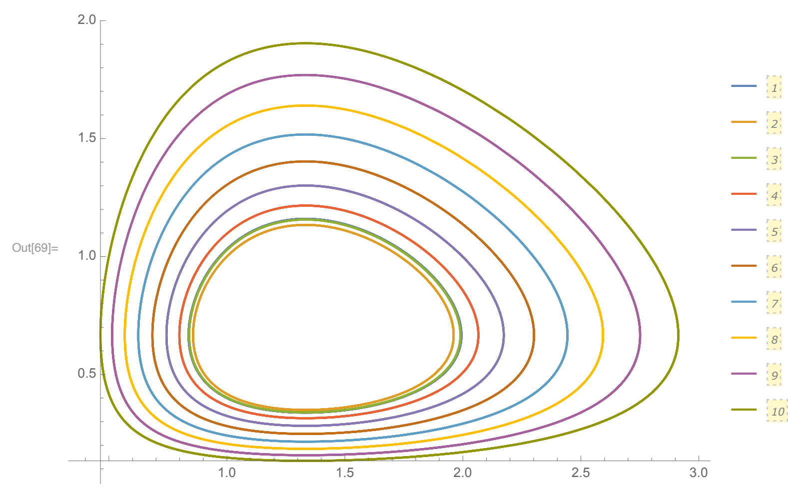
> CHECK YOUR UNDERSTANDING
> Show that
>
> $$cy - s\log(y) = b\log(x) - rx + C, \quad \text{for a constant } C.$$

The above equation gives an *implicit* description of the relationship between y and x: we can rearrange to obtain

$$x^b e^{-rx} = Ae^{cy} y^s \tag{4.3}$$

where A is a constant.

In general, **it's impossible to isolate y in the above equation**. However, numerical methods can be used to determine the curve described by (4.3). Graphs of some particular solutions (i.e. for different values of A) for the Lotka-Volterra system $b = \frac{4}{3}$, $r = 1$, $s = 2/3$, $c = 1$, are shown below:



Remark 4.1. Let x denote the population of foxes (in thousands) and y denote the population of rabbits (in thousands) in a closed system. The graphs plotted above show that as x (the rabbit population) increases this spurs an initial slow growth in y (the fox population). Once the number of rabbits is approx. 3000 the rabbit population begins to decrease, while the fox population keeps growing at a steady rate. However, once the population of foxes reaches 2000 there are too few rabbits leading to a sharp decrease in the fox population.

The 'centre point' about which the graphs are situated is the *stable equilibrium of the system:* if the system begins with approx. 666 foxes and 1333 rabbits then the populations are in a relative stable equilbrium.