Middlebury College

Calculus II: Fall 2017

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.5.
- *Calculus II*, Marsden, Weinstein: Chapter 8.6.

Differential Equations IV

1 Linear First-Order Equations In this paragraph we introduce a substantially more general class of first-order differential equations, known as *linear first-order differential equations*.

Linear First-Order Equations

A **linear first-order differential equation** is a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (*)$$

where $P(x), Q(x)$ are continuous functions defined on some common interval.

Check your understanding

Which of the the following equations are linear first-order equations?

$$(A) : ;\ \frac{dy}{dx} = xy^2 + x, \quad (B) : \ \frac{dy}{dx} = g(x)h(y), \quad (C) : \ \frac{dy}{dx} = ky, \quad (D) : \ x^2y' + xy = 1$$

Let's consider how we may solve a linear first-order equation. We use the following **nifty trick**: we want to solve the general linear first-order equation $(*)$. Define

$$z = y \exp\left(\int P(x)dx\right)$$

Then, $z = z(x)$ and

$$\frac{dz}{dx} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Hence, we find

$$z = \underline{\hspace{5cm}}$$

$$\Rightarrow y = \underline{\hspace{7cm}}$$

In summary,

Method for Solving Linear First-Order Equations

(1) Compute an antiderivative of $P(x)$ i.e. determine $\int P(x)dx$ (no constant of integration required)

(2) Let $I(x) = \exp\left(\int P(x)dx\right)$. Multiply both sides of $(*)$ by $I(x)$.

(3) In this way $(*)$ becomes

$$\frac{d}{dx}\left(I(x)y\right) = Q(x)I(x) \qquad (***)$$

(4) Integrate both sides of $(***)$

(5) Rearrange for y.

Definition 1.1. Given a linear first-order equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we define the **integrating factor** to be

$$I(x) = \exp\left(\int P(x)dx\right)$$

Here $\int P(x)dx$ is *some* antiderivative of $P(x)$ (no constant of integration required).

Example 1.2. 1. Consider the linear first-order equation

$$\frac{dy}{dx} + 3x^2y = 6x^2$$

Here $P(x) =$ ________, $Q(x) =$ ________. Then, we can determine an antiderivative

$$\int P(x)dx = \underline{\hspace{4cm}}$$

and the integrating factor is $I(x) =$ __________.

Multiplying both sides of the differential equation by $I(x)$ gives

$$\underline{\hspace{4cm}} = \underline{\hspace{2.5cm}}$$

You can (and should) check that

$$\frac{d}{dx}\left(\underline{\hspace{2.5cm}}\right) = \underline{\hspace{4cm}}$$

Integrating both sides gives

$$\underline{\hspace{2.5cm}} = \underline{\hspace{5cm}}$$

Rearranging , we obtain the general solution

$$y = \underline{\hspace{4cm}}$$

2. Consider the differential equation

$$x^2y' + xy = 1, \qquad x > 0$$

This can be put into the form of a linear first-order equation by dividing through by x^2:

$$y' + \frac{1}{x}y = \frac{1}{x^2} \tag{$\bullet$}$$

Here $P(x) = \frac{1}{x}$, $Q(x) = \frac{1}{x^2}$. Then, the integrating factor is

$$I(x) = \exp\left(\int P(x)dx\right) = \exp(\log(x)) = x.$$

Here we have used that $x > 0$ to ensure that $\log(x)$ is well-defined. Hence, multiplying ($\bullet$) by $I(x)$ gives

$$xy' + y = \frac{1}{x} \quad \text{or} \quad (xy)' = \frac{1}{x}$$

Integrating both sides we find

$$xy = \int \frac{1}{x}dx = \log(x) + C \quad \Rightarrow \quad y = \frac{\log(x) + C}{x}$$

Given the initial-value $y(1) = 2$, we compute

$$2 = y(1) = \frac{\log(1) + C}{1} = C.$$

Therefore, the solution to ($\bullet$) satisfying $y(1) = 2$ is

$$y = \frac{\log(x) + 2}{x}$$

Check your understanding

1. Solve the initial-value problem

$$y' = x - y, \quad y(0) = 3$$

2. Newton's Second Law states that the force acting upon an object is equal to ma, where a is the acceleration of the body and m is its mass. It is a fact that $a = \frac{dv}{dt}$, where $v = v(t)$ is the velocity of the object at time t.

   A falling object is being acted upon by the downward force of gravity ($= mg$, where g is the gravitational constant) and an opposite force of air resistance ($= kv$, where k is a constant measuring air friction). Hence, using Newton's Second Law, the veolicty of a falling object in a resisted medium is described by the differential equation

$$m\frac{dv}{dt} = mg - kv$$

   If an object is released from rest, determine its velocity $v(t)$. What happens as $t \to \infty$?

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

Determine a solution to the following

1. Solve the intial-value problem

$$\frac{dy}{dx} = y\cos(x) + 2\cos(x), \quad y(0) = 0$$

2. Determine the general solution

$$y' = x^3 y - x^3$$

3. Solve the initial-value problem

$$xy' = \exp(x) - y, \quad y(1) = 0$$