Middlebury College

Calculus II: Fall 2017

**Contact:** `gmelvin@middlebury.edu`

# November 1 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 8.2.
- *Integral Calculus*, Khan Academy: Volumes using calculus.

## Applications of integration II: Definite integration, a reminder

In this lecture we will recall definite integration. We will see how the techniques of integration we have developed (e.g. method of integration by parts, method of substitution etc.) are affected when we want to compute a definite integral.

**1 Definite integration** The Fundamental Theorem of Calculus tells us a solution to an **antiderivative problem**: let $f(x)$ be a continuous function defined on $[a, b]$. Then, the function

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$, namely,

$$\frac{d}{dx}F(x) = f(x).$$

Moreover, we've seen that, if $G(x)$ is another antiderivative of $f(x)$, so that

$$\frac{d}{dx}G(x) = f(x),$$

then there is a numerical constant $C$ such that

$$F(x) = G(x) + C.$$

Inputting $x = a$ into both sides of the above equation gives

$$\underline{\qquad\quad} = F(a) = G(a) + C \quad \Longrightarrow \quad C = \underline{\qquad\qquad\quad}.$$

Hence,

$$F(x) = \underline{\qquad\qquad\qquad\quad}$$

This gives the following Corollary of the Fundamental Theorem of Calculus:

> Let $f(x)$ be a continuous function defined on $[a, b]$. Let $G(x)$ be an antiderivative of $f(x)$. Then,
>
> $$\int_a^b f(x)dx = G(b) - G(a)$$

**Remark 1.1.** The left hand side of the above formula is called a **definite integral**: it is the (signed) area between the graph of $f(x)$ and the $x$-axis, where $a \leq x \leq b$. The result states that we can compute this area by the following method:

1. Solve the antiderivative problem $\frac{d}{dx}F(x) = f(x)$;
2. Compute $F(b) - F(a)$.

Let's think about how definite integration (i.e. determining *area*) is affected by the techniques we've developed to solve antiderivative problems.

## Method of substitution

Check your understanding

Consider the function $h(x) = x\sqrt{1+x^2}$, defined for all $x$.

1. Fill in the blanks to determine an antiderivative $H(x)$ of $h(x)$.

   Let $u =$ ______________, so that $\frac{du}{dx} =$ ______________.

   Since
   $$x\sqrt{1+x^2} = \underline{\hspace{4cm}}\frac{du}{dx}$$
   the method of substitution gives
   $$H(x) = \int x\sqrt{1+x^2}dx = \int \underline{\hspace{5cm}}du$$
   $$= \underline{\hspace{8cm}}$$
   $$= \underline{\hspace{8cm}}$$

2. Find $f(x)$ and $g(x)$ so that $H(x) = f(g(x))$. (*Hint: look at the previous problem*)
   $$f(x) = \underline{\hspace{3cm}} \qquad g(x) = \underline{\hspace{3cm}}$$

3. Given that
   $$\int_1^3 h(x)dx = H(3) - H(1)$$
   determine $a, b$ such that
   $$\int_a^b f'(u)du = H(3) - H(1)$$
   (*Hint: what is an antiderivative of $f'(u)$? Can you see how to use the previous problems?*)

## Method of inverse trigonometric substitution

Let's see how definite integration is affected by the method of inverse trigonometric substitutionn

CHECK YOUR UNDERSTANDING

Let $r > 0$. Consider the function $f(x) = \sqrt{r^2 - x^2}$, defined for $-r \leq x \leq r$.

1. Fill in the blanks to determine an antiderivative $F(x)$ of $f(x)$.

   Let $x =$ ______________, so that $\frac{dx}{dt} =$ ______________.

   Since
   $$\sqrt{r^2 - x^2}\frac{dx}{dt} = \rule{4cm}{0.4pt}$$
   the method of substitution gives
   $$F(x) = \int \sqrt{r^2 - x^2}dx = \int \rule{5cm}{0.4pt} dt$$
   $$= \rule{9cm}{0.4pt}$$
   $$= \rule{9cm}{0.4pt}$$

2. Given that
   $$\int_0^r f(x)dx = F(r) - F(0)$$
   determine $a, b$ such that
   $$\int_a^b f(x(t))dt = F(r) - F(0)$$
   (*Hint: can you see how to use the approach you determined on p.2* in reverse*? You'll need an appropriate inverse function.*)

3. Let $A$ be the area of a circle having radius $r > 0$. Explain why
   $$A = 4\int_0^r \sqrt{r^2 - x^2}dx$$
   and deduce the well-known formula for $A$.

## Method of integration by parts

Let's see how definite integration is affected by the method of integration by parts.

Check your understanding

Let $f(x) = x$ and $g(x) = \log(x)$.

1. Explain why

$$\int_1^2 (\log(x) + 1)\, dx = f(2)g(2) - f(1)g(1)$$

2. Use the previous problem to complete the following statement:

> Let $f(x)$, $g(x)$ be differentiable functions. Then,
>
> $$\int_a^b f(x)g'(x)dx = \underline{\hspace{6cm}} - \int_{\_\_}^{\_\_} \underline{\hspace{4.5cm}}dx$$

**Example 1.2.** 1. Recall the formula for the arc length $L$ of $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$:

$$L = \int_{-2}^{2} \sqrt{1 + 4x^2}dx = 2\int_0^2 \sqrt{1 + 4x^2}dx$$

Let $x = \frac{1}{2}\tan(t)$, $\frac{dx}{dt} = \frac{1}{2}\sec^2(t)$. Then, when

$$x = 0, \quad t = \underline{\hspace{4.5cm}}$$

$$x = 2, \quad t = \underline{\hspace{4.5cm}}$$

As

$$\sqrt{1 + 4x^2}\frac{dx}{dt} = \underline{\hspace{7.5cm}}$$

the method of inverse trig. substitution gives

$$L = \int_{t=0}^{t=\arctan(4)} \sec^3(t)dt$$

Using integration by parts, and the fact that

$$\frac{d}{dt}\log(\sec(t) + \tan(t)) = \sec(t),$$

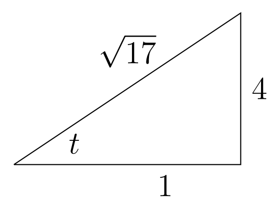
it can be shown (try it yourself!) that an antiderivative of $\sec^3(t)$ is

$$\frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\log|\sec(t)+\tan(t)|$$

Hence,

$$L = \int_{t=0}^{t=\arctan(4)} \sec^3(t)dt = \left[\frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\log|\sec(t)+\tan(t)|\right]_{t=0}^{t=\arctan(4)}$$

If $4 = \tan(t)$ then we can use the triangle




to compute

$$L = 2\sqrt{17} + \frac{1}{2}\log(\sqrt{17}+4)$$

2. Using integration by parts we can determine

$$\int_0^{\pi/2} \cos^3(x)dx$$

as follows: let

$$\begin{array}{ll} f = \cos^2(x) & g' = \cos(x) \\ f' = -2\sin(x)\cos(x) & g = \sin(x) \end{array}$$

so that, by the method of integration by parts,

$$\int_0^{\pi/2} \cos^3(x)dx = \left[\cos^2(x)\sin(x)\right]_0^{\pi/2} + 2\int_0^{\pi/2} \sin^2(x)\cos(x)dx = 2\int_0^{\pi/2} \sin^2(x)\cos(x)dx$$

since $\sin(0) = \cos(\pi/2) = 0$. Let $u = \sin(x)$, $\frac{du}{dx} = \cos(x)$. Then,

$$x = 0 \quad \Longrightarrow \quad u = \underline{\hspace{3cm}}$$

$$x = \pi/2 \quad \Longrightarrow \quad u = \underline{\hspace{3cm}}$$

Since

$$\sin^2(x)\cos(x) = u^2\frac{du}{dx}$$

the method of substitution gives

$$2\int_0^{\pi/2} \sin^2(x)\cos(x)dx = \int \underline{\hspace{6cm}}$$

Hence,

$$\int_0^{\pi/2} \cos^3(x)dx = \underline{\hspace{6cm}}$$

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problem. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

1. Determine the definite integral $\int_1^2 \frac{\log(x)}{x} dx$.
2. Determine the definite integral $\frac{2}{\pi}\int_0^{\pi/2} x\sin(x)dx$.
3. Determine the arc length of $y = 2\sqrt{x}$ between $x = 0$ and $x = 1$.
4. Challenge! Use integration by parts to show that

$$\int \sec^3(t)dt = \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\log(\sec(t) + \tan(t)) + C$$