Middlebury College

Calculus II: Fall 2017

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.1, 9.3.
- *Calculus II*, Marsden, Weinstein: Chapter 8.2.

Differential Equations III

1 Examples of Growth & Decay In this paragraph we consider some examples of growth and decay.

Natural growth and decay

The general solution to the **Growth & Decay Equation**

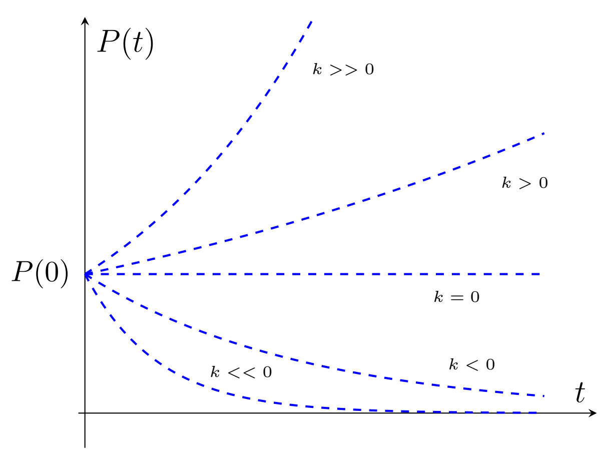
$$\frac{dP}{dt} = kP, \qquad k \text{ constant},$$

is given by

$$P(t) = C\exp(kt),$$

for some constant C. Moreover, given any real number C_0 there exists ____________________ solution satisfying $P(0) = C_0$.

The relationship between the sign of k and the behaviour of $P(t)$ is shown below.



Example 1.1 (**Newton's Law of Cooling**). Newton's Law of Cooling states the following:

The instantaneous rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature of its surroundings.

Let

$T(t) =$ temperature of object at time t.
$T(0) = T_0 =$ initial temperature of the object.
$T_a =$ ambient temperature of surrounding.

Newton's Law of Cooling states that there is a constant k such that

$$\frac{dT}{dt} = k(T - T_a).$$

Define

$y(t) = T(t) - T_a =$ temperature difference between object and surroundings at time t..
$y(0) = T_0 - T_a =$ initial temperature difference at $t = 0$.

Then, since T_a is a constant, we can rewrite the above differential equation as

$$\frac{dy}{dt} = ky$$

Hence, the rate at which an object cools relative to its surrounding is governed by a Growth & Decay Equation. In particular, its general solution is

$$y(t) = y_0 \exp(kt)$$

Hence, the temperature of the object is

$$T(t) = y(t) + T_a = (T_0 - T_a)\exp(kt) + T_a$$

2 Separable equations The Growth & Decay Equation is a particular example of a class of differential equations known as *separable equations*.

> If a differential equation can be written in the form
>
> $$\frac{dy}{dx} = g(x)h(y) \qquad (**)$$
>
> where the right hand side factors as the product of a function of x and a function of y then it is called a **separable equation**.

Check your understanding

Which of the following are examples of separable equations?

$$(A): \frac{dy}{dx} = \frac{-3x}{y + yx^2}, \quad (B): \frac{dz}{dx} = z^2 + 2zx + x^2, \quad (C): \frac{dP}{dt} = kP, \quad (D): yy' = \cos(2x)$$

The approach to solving separable equations - i.e. determining all solutions $y = y(x)$ satisfying $(**)$ - is guided by our solution to the GDE: if we rearrange $(**)$ to get

$$\frac{1}{h(y)} \cdot \frac{dy}{dx} = g(x)$$

then the method of inverse substitution gives

$$\int \frac{1}{h(y)} dy = \int g(x) dx + C$$

Solutions to Separable Differential Equations

Given a separable differential equation

$$\frac{dy}{dx} = g(x)h(y)$$

1. Solve the antiderivative problems

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

2. Solve for y, if possible.

Example 2.1. 1. Consider the separable equation

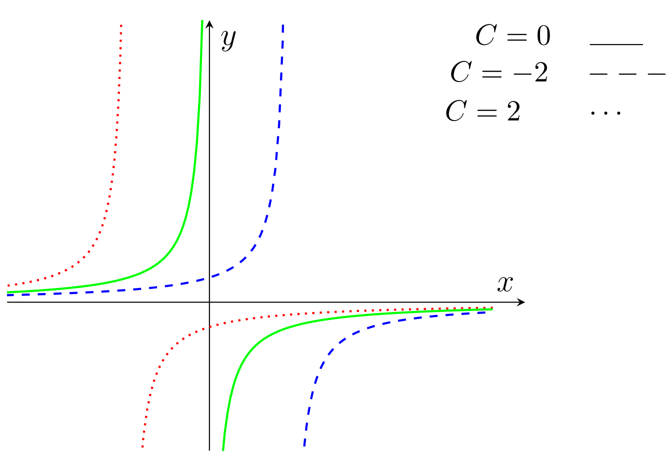
$$\frac{dy}{dx} = y^2$$

Here $h(y) = y^2$ and $g(x) = 1$. Then, we find

$$-\frac{1}{y} = \int \frac{1}{y^2} dy = \int dx = x + C$$

so that

$$y = \frac{1}{-C - x}$$



2. Consider the separable equation

$$\frac{dy}{dx} = \frac{-3x}{y + yx^2}.$$

Here $g(x) = \frac{-3x}{1+x^2}$ and $h(y) = \frac{1}{y}$. Then, we find

$$\frac{y^2}{2} = \int y dy = \int \frac{-3x}{1+x^2} dx = -\frac{3}{2}\log(1+x^2) + C$$

If we are looking for the solution satisfying $y = 2$ when $x = 0$, then we must have

$$\frac{4}{2} = -\frac{3}{2}\log(1+0) + C \qquad \Longrightarrow \qquad C = 2$$

Therefore, y must satisfy

$$y^2 = -3\log(1+x^2) + 4$$

There are two possible choices for y as a function of x, given by a choice of square root. Since we require y is positive nearby to $x = 0$ (i.e. y is near to 2 when x is near to 0) then we choose the positive square root and

$$y = \sqrt{4 - 3\log(1+x^2)}$$

Observe that $y = y(x)$ is only defined whenever x satisfies

$$\frac{4}{3} \geq \log(1+x^2) \quad \Leftrightarrow \quad e^{4/3} \geq 1 + x^2 \quad \Leftrightarrow \quad 1 - e^{4/3} \leq x \leq e^{4/3} - 1$$

Check your understanding

1. Determine the solution to the separable differential equation

$$\frac{dy}{dx} = x^2 \exp(-y)$$

2. (a) Write down a separable differential equation that describes a curve whose slope at (x, y) is xy.

(b) Find an equation of the curve passing through the point $(1, 0)$ and whose slope at (x, y) is xy.

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

Determine a solution to the following

1. Solve the GDE and sketch the graph of the unique solution.

$$\frac{dx}{dt} - 3x = 0, \quad x(0) = 1$$

2. Solve the equation for $f(t)$ and sketch its graph.

$$f'(t) + 2f(t) = 0, \quad f(0) = 1.$$

3. Solve the separable differential equation

$$y\frac{dy}{dx} = x, \quad y(0) = 1.$$