Middlebury College

Calculus II: Fall 2017

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.1, 9.3.
- *Calculus II*, Marsden, Weinstein: Chapter 8.2.

Differential Equations II

We complete our discussion of *growth and decay differential equations*.

1 Solutions of the growth & decay equations

Recall the

Growth & Decay Equation (GDE)

Let $P(t)$ be the number of individuals in a population of interest at time t. Then, the basic growth/decay equation governing $P(t)$ is

$$\frac{dP}{dt} = kP$$

- If $k > 0$ then $P(t)$ is **growing**.
- If $k < 0$ then $P(t)$ is **decaying**.

This equation provides a basic model for quantities - for example, radioactive substances, bank balances, temperature of objects - that grow/decay continuously at a rate proportional to their current value.

Basic Assumption: We will assume that $P(t) \neq 0$, for all t.

Remark 1.1. The growth & decay equation given above describes the growth/decay of populations, *in the absence of external factors*. As such, this equation is not a completely accurate model of most growth/decay situations.

For example, suppose that a population initially grows at a constant rate proportional to $P(t)$, but begins to decay once $P(t)$ passes a certain threshold M (the *carrying capacity of* $P(t)$)

- $\frac{dP}{dt} = kP$ if P is small.
- $\frac{dP}{dt} < 0$ if $P > M$ (i.e. P decreases once it exceeds M).

A simple expression that incorporates both assumptions is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad k > 0.$$

Indeed, if P is small compared to M then P/M is small, while if $P > M$ then $1 - \frac{P}{M} < 0$ so that $\frac{dP}{dt} < 0$. The differential equation described above is known as the **logistic differential equation (LDE)**.

More generally, the modification of the GDE by additional factors is called *damping*.

In order to solve the GDE - i.e. to find all solutions $P(t)$ satisfying the given GDE - we **recognise** that it looks like a piece of an antiderivative problem whose solution is obtained by the **method of inverse substitution**: by our Basic Assumption, we may rearrange the GDE to obtain

$$\frac{1}{P} \cdot \frac{dP}{dt} = k$$

Writing $P = P(t)$ as a function of t, the method of inverse substitution gives

$$\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_.$$

Evaluating the two antiderivative problems above gives

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

and applying exp to both sides gives

$$P(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Observe that we can determine

____

Check your understanding

Let $P(t), Q(t)$ be two solutions to the GDE

$$\frac{dP}{dt} = -2P \qquad (*)$$

Assume that both functions $P(t), Q(t)$ are not identically zero.

1. Fill in the blanks:
$$P'(t) = \_\_\_\_\_\_\_\_ \qquad Q'(t) = \_\_\_\_\_\_\_\_$$
2. Define $R(t) = \frac{P(t)}{Q(t)}$. Use the Chain Rule and $(*)$ to show that $R'(t) = 0$.

3. Use the previous problem to explain why there is a constant a satisfying $P(t) = aQ(t)$, for all t.

4. Suppose that $P(0) = Q(0) \neq 0$. Explain why this condition implies $P(t) = Q(t)$, for all t.

5.

> Let C_0 be a real number. There __________________________ solution(s) $P(t)$ of the GDE
>
> $$\frac{dP}{dt} = kP,$$
>
> satisfying $P(0) = C_0$.

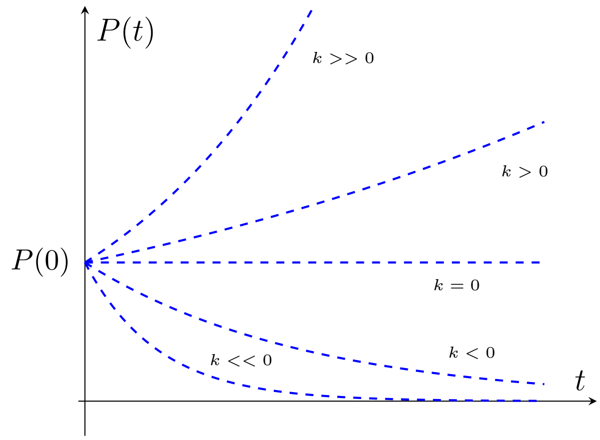
Circle the phrase that completes the statement above.

does not exist any	**exists at least two distinct**	**is a unique**

Natural growth and decay

> The general solution to the **Growth & Decay Equation**
>
> $$\frac{dP}{dt} = kP, \qquad k \text{ constant},$$
>
> is given by
>
> $$P(t) = C\exp(kt),$$
>
> for some constant C. Moreover, given any real number C_0 there exists _____________________ solution satisfying $P(0) = C_0$.

The relationship between the sign of k and the behaviour of $P(t)$ is shown below.



Example 1.2 (Newton's Law of Cooling). Newton's Law of Cooling states the following:

> *The instantaneous rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature of its surroundings.*

Let

$T(t) =$ temperature of object at time t.
$T(0) = T_0 =$ initial temperature of the object.
$T_a =$ ambient temperature of surrounding.

Newton's Law of Cooling states that there is a constant k such that

$$\frac{dT}{dt} = k(T - T_a).$$

Define

$y(t) = T(t) - T_a =$ temperature difference between object and surroundings at time t..
$y(0) = T_0 - T_a =$ initial temperature difference at $t = 0$.

Then, since T_a is a constant, we can rewrite the above differential equation as

$$\frac{dy}{dt} = ky$$

Hence, the rate at which an object cools relative to its surrounding is governed by a Growth & Decay Equation. In particular, its general solution is

$$y(t) = y_0 \exp(kt)$$

Hence, the temperature of the object is

$$T(t) = y(t) + T_a = (T_0 - T_a)\exp(kt) + T_a$$

2 Separable equations

The Growth & Decay Equation is a particular example of a class of differential equations known as *separable equations*.

If a differential equation can be written in the form

$$\frac{dy}{dx} = g(x)h(y) \qquad (**)$$

where the right hand side factors as the product of a function of x and a function of y then it is called a **separable equation**.

CHECK YOUR UNDERSTANDING

Which of the following are examples of separable equations?

$$(A): \frac{dy}{dx} = \frac{-3x}{y + yx^2}, \quad (B): \frac{dz}{dx} = z^2 + 2zx + x^2, \quad (C): \frac{dP}{dt} = kP, \quad (D): yy' = \cos(2x)$$

The approach to solving separable equations - i.e. determining all solutions $y = y(x)$ satisfying $(**)$ - is guided by our solution to the GDE: if we rearrange $(**)$ to get

$$\frac{1}{h(y)} \cdot \frac{dy}{dx} = g(x)$$

then the method of inverse substitution gives

$$\int \frac{1}{h(y)} dy = \int g(x) dx + C$$

Example 2.1. 1. Consider the separable equation

$$\frac{dy}{dx} = y^2$$

Here $h(y) = y^2$ and $g(x) = 1$. Then, we find

$$-\frac{1}{y} = \int \frac{1}{y^2} dy = \int dx = x + C$$

so that

$$y = \frac{1}{-C - x}$$

2. Consider the separable equation

$$\frac{dy}{dx} = \frac{-3x}{y + yx^2}.$$

Here $g(x) = \frac{-3x}{1+x^2}$ and $h(y) = \frac{1}{y}$. Then, we find

$$\frac{y^2}{2} = \int y dy = \int \frac{-3x}{1 + x^2} dx = -\frac{3}{2} \log(1 + x^2) + C$$

If we are looking for the solution satisfying $y = 2$ when $x = 0$, then we must have

$$\frac{4}{2} = -\frac{3}{2}\log(1+0) + C \qquad \Longrightarrow \qquad C = 2$$

Therefore, y must satisfy

$$y^2 = -3\log(1+x^2) + 4$$

There are two possible choices for y as a function of x, given by a choice of square root. Since we require y is positive nearby to $x = 0$ (i.e. y is near to 2 when x is near to 0) then we choose the positive square root and

$$y = \sqrt{4 - 3\log(1+x^2)}$$

Observe that $y = y(x)$ is only defined whenever x satisfies

$$\frac{4}{3} \geq \log(1+x^2) \quad \Leftrightarrow \quad e^{4/3} \geq 1 + x^2 \quad \Leftrightarrow \quad 1 - e^{4/3} \leq x \leq e^{4/3} - 1$$

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

Determine a solution to the following

1. Solve the GDE and sketch the graph of the unique solution.
$$\frac{dx}{dt} - 3x = 0, \quad x(0) = 1$$
2. Solve the equation for $f(t)$ and sketch its graph.
$$f'(t) + 2f(t) = 0, \quad f(0) = 1.$$
3. Solve the separable differential equation
$$y\frac{dy}{dx} = x, \quad y(0) = 1.$$