Middlebury College

Calculus II: Fall 2017

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.10, 11.11
- *Power Series*, Integral Calculus, Khan Academy

Taylor Series II

In this lecture we conclude our discussion of power and Taylor series of infinitely differentiable functions. We will see applications of Taylor polynomials to approximate functions.

1 Taylor Series Let $f(x)$ be an infinitely differentiable function. If $f(x)$ is the limit of its Taylor series (centred at c) then $f(x) = \lim_{n\to\infty} T_n(x)$. Let

$$R_n(x) = f(x) - T_n(x),$$

the **remainder** of the Taylor series. We have the following observation:

> If $\lim_{n\to\infty} R_n(x) = 0$ whenever $|x-c| < R$ then $\lim_{n\to\infty} T_n(x) = f(x)$ on the interval $|x-c| < R$.

The following result provides us with a tool to determine the behaviour of $R_n(x)$ as $n \to \infty$.

Taylor's Theorem/Inequality

> If $|f^{(n+1)}(x)| \leq M$ for $|x-c| \leq d$ then
>
> $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1} \qquad \text{for } |x-c| \leq d$$
>
> In particular, whenever $|x-c| \leq d$ we have
>
> $$|R_n(x)| \leq \frac{Md^{n+1}}{(n+1)!}$$

If we can find a constant M and natural number N with the property that

$$|f^{(n+1)}(x)| \leq M, \quad \text{for any } n \geq N \text{ and } |x-c| \leq d$$

then $\lim |R_n(x)| = 0$. Hence, in this situation, $f(x)$ **equals its Taylor series on the interval** $[c-d, c+d]$.

Reminder: For any real number $c > 0$, $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

Example 1.1. Let $f(x) = \sin(x)$. Then, since any derivative of $f(x)$ is either equal to $\pm\sin(x)$ or $\pm\cos(x)$, we have

$$|f^{(n)}(x)| \leq 1, \quad \text{for any } n = 0, 1, 2, 3, \ldots, \text{ and any } x.$$

Take, for example, $d = 10$ (this is an arbitrary choice). Then, for any n, we have

$$|f^{(n+1)}(x)| \leq 1 \quad \text{whenever } |x| \leq 10.$$

Hence, Taylor's Inequality implies that

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \leq \frac{10^{n+1}}{(n+1)!} \quad \text{for } |x| \leq 10 \text{ and any } n.$$

This means

$$\underline{\hspace{3cm}} \leq R_n(x) \leq \underline{\hspace{3cm}} \qquad \text{for } |x| \leq 10.$$

Check your understanding

Complete the following statement:

By the ______________ Theorem, we conclude that $\lim_{n\to\infty} R_n(x) =$ __________, whenever $|x| \leq 10$.

Hence, for any x in the interval ___________________ we have

$$\sin(x) = \underline{\hspace{8cm}}$$

Since d was arbitrary we obtain the following series representation of $\sin(x)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{for any } x.$$

A similar argument shows that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{for any } x.$$

Check your understanding

1. Show that the series
$$1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots$$
is convergent and determine its limit.

2. Show that the series
$$1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots$$
is convergent and determine its limit.

3. Let $f(x) = \exp(x)$ and $d > 0$. Determine a constant K such that $|f^{(n)}(x)| \leq K$, for any n and any $|x| \leq d$.

4. Using Taylor's Inequality, deduce that the Taylor series of $f(x) = \exp(x)$ centred at $c = 0$ equals $f(x)$, for all x.