Middlebury College

Calculus II: Fall 2017

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.10, 11.11
- *Power Series*, Integral Calculus, Khan Academy

Taylor Series

In this lecture we will investigate the conditions that a function must satisfy in order for it to possess a representation by power series (centred at c): this is the content of *Taylor's Theorem*.

1 Taylor Series We are interested in the following problem.

> **Problem:** Let $f(x)$ be a given function.
>
> 1. Does $f(x)$ admit a power series representation? If so, how do we determine it?
> 2. For which x does the power series representation make sense?

In the last lecture, we investigated how to determine the coefficients of a power series representation

Definition 1.1. Let $f(x)$ be an infinitely differentiable function. The **Taylor series of $f(x)$ centred at c** is the series

$$\sum_{n=0}^{\infty} \underline{\hspace{3cm}}(x-c)^n = \underline{\hspace{8cm}}$$

When $c = 0$ the Taylor series associated to $f(x)$ is also called the **Maclaurin series of $f(x)$** (after the Scottish mathematician Colin Maclaurin (1698-1746))

Remark 1.2 (IMPORTANT!)**.** At this time, the Taylor series of $f(x)$ centred at c is a series that we are **associating** to $f(x)$; we are **not** saying that the Taylor series is equal to $f(x)$. We will investigate when the Taylor series equals $f(x)$ in the next lecture.

Example 1.3. 1. Let $f(x) = \sin(x)$. Then, $f(x)$ is an infinitely differentiable function and we can determine its associated Taylor series at $c = 0$ (i.e. the Maclaurin series). We compute

$$f(0) = \underline{\hspace{1cm}}$$

$$f'(0) = \underline{\hspace{1cm}}$$

$$f''(0) = \underline{\hspace{1cm}}$$

$$\vdots$$

In general

$$f^{(n)}(0) = \begin{cases} \_\_\_\_\_, & \text{if } n \text{ even,} \\ \\ \_\_\_\_\_, & \text{if } n = 2k+1. \end{cases}$$

Hence, the Taylor series associated to $f(x) = \sin(x)$ at $c = 0$ is

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CHECK YOUR UNDERSTANDING

For each $f(x)$, determine the first five terms of the associated Taylor series at c.

1. $f(x) = \cos(x)$, $c = 0$:

2. $f(x) = \sin(x)$, $c = \pi$:

We consider the following question:

Question: Under what circumstances is a function equal to the sum of its Taylor series (centred at c) i.e. when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{j!}(x-c)^n \ ?$$

This holds precisely when $f(x)$ is the limit of the partial sums of its associated Taylor series.

Definition 1.4. Let $f(x)$ be an infinitely differentiable function defined on some interval containing $x = c$. The n^{th}**-degree Taylor polynomial of** f **centred at** c is

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{j=0}^{n} \frac{f^{(j)}(c)}{j!}(x-c)^j$$

Remark 1.5. Observe that the n^{th}-degree Taylor polynomial $T_n(x)$ is precisely the n^{th} partial sum of the Taylor series associated to f, centred at c.

Example 1.6. 1. The n^{th}-degree Taylor polynomial of $f(x) = \exp(x)$ centred at $c = 0$ is

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

2. The 5^{th}-degree Taylor polynomial of $f(x) = \sqrt{1+x}$ centred at $c = 0$ is

$$\begin{aligned} T_5(x) = 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{x^2}{2!} + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{x^3}{3!} + \ldots \\ + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\frac{x^4}{4!} + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)\frac{x^5}{5!} \\ = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \frac{15}{184}x^4 + \frac{105}{3840}x^5 \end{aligned}$$

The coefficients are determined by computing successive derivatives of $f(x)$ at $x = 0$. For example, the coefficient of x^3 is determined by computing

$$f'''(x) = \frac{d^3}{dx^3}(1+x)^{1/2} = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}(1+x)^{-5/2} \implies \frac{f'''(0)}{3!} = \frac{1}{3!} \cdot \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} = \frac{3}{48}$$

If $f(x)$ is the limit of its Taylor series then $f(x) = \lim_{n\to\infty} T_n(x)$. Let

$$R_n(x) = f(x) - T_n(x)$$

the **remainder** of the Taylor series.

Check your understanding

Complete the following statement

> If $\lim_{n\to\infty} R_n(x) =$ _____ whenever $|x - c| < R$ then $\lim_{n\to\infty} T_n(x) = f(x)$ on
>
> the interval _______________.

Therefore, if we can show that the remainder $R_n(x)$ gets arbitrarily _______________ as $n \to \infty$, then the Taylor series of f (centred at c) gives a power series representation of $f(x)$.

The following result provides us with a tool to determine the behaviour of $R_n(x)$ as $n \to \infty$.

Taylor's Inequality

If $f^{(n+1)}(x) \leq M$ for $|x - c| \leq d$ then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - c|^{n+1} \quad \text{for } |x - c| \leq d$$

Example 1.7. Let $f(x) = \sin(x)$. Then, since any derivative of $f(x)$ is either equal to $\pm\sin(x)$ or $\pm\cos(x)$, we have

$$|f^{(n)}(x)| \leq \underline{\hspace{3cm}}, \quad \text{for any } n = 0, 1, 2, 3, \ldots$$

Take, for example, $d = 10$ (this is an arbitrary choice). Then, for any n, we have

$$|f^{(n+1)}(x)| \leq 1 \quad \text{whenever } |x| \leq 10.$$

Hence, Taylor's Inequality implies that

$$|R_n(x)| \leq \underline{\hspace{4cm}} \leq \underline{\hspace{4cm}} \quad \text{for } |x| \leq 10.$$

This means

$$\underline{\hspace{4cm}} \leq R_n(x) \leq \underline{\hspace{4cm}} \quad \text{for } |x| \leq 10.$$

CHECK YOUR UNDERSTANDING

Complete the following statement:

By the ______________ Theorem, we conclude that $\lim_{n\to\infty} R_n(x) =$ __________, whenever $|x| \leq 10$.

Hence, for any x in the interval __________________ we have

$$\sin(x) = \underline{\hspace{8cm}}$$

GET CREATIVE!

In what way did our above investigation depend on the choice $d = 10$ (if at all)? Determine all x for which the Taylor series of $f(x) = \sin(x)$ (centred at $c = 0$) equals $f(x)$ and complete the following statement:

The Taylor series expansion

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

holds ________________________

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

- Determine the Taylor series associated to $f(x)$ at c.

  1. $f(x) = \frac{1}{(1-x)^2}$, $c = 0$.
  2. $f(x) = x^4 - 3x^2 + 1$, $c = 1$
  3. $f(x) = \ln(x)$, $c = 2$

- Show that $f(x)$ is equal to its Taylor series centred at c on the given interval I.

  1. $f(x) = \cos(x)$, $c = 0$, $I = (-\infty, \infty)$.
  2. $f(x) = \exp(x)$, $c = 0$, $I = (-10, 10)$.
  3. $f(x) = \exp(x)$, $c = 0$, $I = (-10^{20}, 10^{20})$.