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December 4 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Power Series*, Integral Calculus, Khan Academy

Power Series III

In this lecture we will continue our analysis of power series representations of functions. Last time we saw how to represent certain functions as power series; this time we will determine when *any* given function admits a representation by power series.

1 Power series representing functions Given a power series $\sum_{n=0}^{\infty} c_n(x-c)^n$ with interval of convergence I, we can define a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-c)^n, \quad \text{for } x \text{ in } I.$$

Any function defined in this way as a power series admits the following properties:

Properties of functions defined by power series:

- $f(x)$ is **differentiable** (and therefore continuous) on I and

$$f'(x) = c_1 + 2c_2(x-c) + 3c_3(x-c)^2 + \dots$$

i.e. the power series can be differentiated term-by-term.

- $f(x)$ is **integrable** and

$$\int f(x)dx = C + c_0(x-c) + \frac{c_1}{2}(x-1)^2 + \frac{c_2}{3}(x-c)^3 + \dots$$

i.e. the power can be integrated term-by-term.

Moreover, the series obtained by differentiation/integration are centred at c and have the same radius of convergence as $f(x)$. The endpoints of the interval of convergence need to be given further investigation.

These results will be very useful in giving power series representations of well-known functions.

Example 1.1. By the Geometric Series Theorem, the function

$$f(x) = \frac{1}{1-x}, \quad -1 < x < 1$$

can be represented as the power series (centred at $c = 0$)

$$f(x) = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Remark 1.2. It is **very important to remember** that a representation of a function by a power series, centred at c, is only valid on the interval of convergence. For example, the function

$$f(x) = \frac{1}{1-x}$$

is defined at $x = 2$, and $f(-1) = \frac{1}{2}$ and $f(2) = -1$. However, the associated power series

$$1 + 2 + 2^2 + 2^3 + \ldots = \sum_{n=1}^{\infty} 2^n$$

does not converge. In fact, the content of the Geometric Series Theorem tells us for which x this series **does converge**: whenever $|x| < 1$.

Check your understanding

Determine power series representations of the following functions.

1. $f(x) = \frac{2}{3-x}$ (*Hint: rearrange so the series is in the form* $\frac{a}{1-y}$, *where* a *is a constant and* y *is in terms of* x)

2. $f(x) = \frac{1}{(2-x^2)^2}$ (*Hint: recall that* $\frac{d}{dy}\left(\frac{1}{1-y}\right) = \frac{1}{(1-y)^2}$)

2 Taylor Series and representations of arbitrary functions by power series

In the past few examples we've seen how to take the well-known power series representation of $f(x) = \frac{1}{1-x}$, valid whenever $-1 < x < 1$, and try to give power series representations of different functions.

> **Problem:** Let $f(x)$ be a given function.
>
> 1. Does $f(x)$ admit a power series representation? If so, how do we determine it?
>
> 2. For which x does the power series representation make sense?

Since a power series with centre $x = c$ is completely determined by its coefficients $c_0, c_1, c_2, \ldots$, it will be useful to have a way to determine these coefficients explicitly.

Mathematical workout - flex those muscles

Let $f(x)$ be a function defined by the power series

$$\sum_{n=0}^{\infty} c_n (x-c)^n, \quad |x-c| < R,$$

so that $I = (c-R, c+R)$ is the associated interval of convergence.

1. True/False: $c_0 = f(0)$.

2. True/False: $c_1 = f'(c)$.

3. Give a formula for c_2, c_3, c_5 in terms of f. (*Hint: your solution will use derivatives and evaluation of f at some point*)

   $c_2 =$ ______________ $c_3 =$ ______________ $c_5 =$ ______________

4. Spot the pattern! Determine a general expression for c_n using $f(x)$:

   $c_n =$ ______________

Definition 2.1. Let $f(x)$ be an infinitely differentiable function. The **Taylor series of $f(x)$ centred at c** is the series

$$\sum_{n=0}^{\infty} \underline{\hspace{4cm}}(x-c)^n = \underline{\hspace{10cm}}$$

When $c = 0$ the Taylor series associated to $f(x)$ is also called the **Maclaurin series of $f(x)$** (after the Scottish mathematician Colin Maclaurin (1698-1746))

Remark 2.2 (IMPORTANT!). At this time, the Taylor series of $f(x)$ centred at c is a series that we are **associating** to $f(x)$; we are **not** saying that the Taylor series is equal to $f(x)$. We will investigate when the Taylor series equals $f(x)$ in the next lecture.

Example 2.3. 1. Let $f(x) = \sin(x)$. Then, $f(x)$ is an infinitely differentiable function and we can determine its associated Taylor series at $c = 0$ (i.e. the Maclaurin series). We compute

$$f(0) = \_\_\_\_\_$$

$$f'(0) = \_\_\_\_\_$$

$$f''(0) = \_\_\_\_\_$$

$$f'''(0) = \_\_\_\_\_$$

$$\vdots$$

In general

$$f^{(n)}(0) = \begin{cases} \_\_\_\_\_, & \text{if } n \text{ even,} \\ \\ \_\_\_\_\_, & \text{if } n = 2k + 1. \end{cases}$$

Hence, the Taylor series associated to $f(x) = \sin(x)$ at $c = 0$ is

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CHECK YOUR UNDERSTANDING

For each $f(x)$, determine the associated Taylor series at c.

1. $f(x) = \cos(x)$, $c = 0$:

2. $f(x) = \sin(x)$, $c = \pi$:

3. $f(x) = \sqrt{1+x}$, $c = 0$:

Mathematical workout - flex those muscles

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

- Determine the Taylor series associated to $f(x)$ at c.

1. $f(x) = \frac{1}{(1-x)^2}$, $c = 0$.
2. $f(x) = x^4 - 3x^2 + 1$, $c = 1$
3. $f(x) = \ln(x)$, $c = 2$