Calculus II: Fall 2017

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December 1 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Power Series*, Integral Calculus, Khan Academy

Power Series II

In this lecture we will develop our understanding of power series. We will see when power series give well-defined functions, their basic properties and how they can be used to represent well-known functions.

1 Convergence of power series Recall that a **power series centred at** c is a series expression

$$\sum_{n=0}^{\infty} c_n(x-c)^n$$

where $c_0, c_1, c_2, \ldots$ are constants and x is a variable.

Yesterday, we saw an approach to determining a solution to the following problem:

Problem: for which x is a power series a *well-defined function*?

Consider the series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

Then, if we let

$$b_n = \frac{(x-1)^n}{n}$$

we can use the Ratio Test to determine the convergence of the series: we computed

$$L = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = |x-1|$$

and the power series is convergent if

$$L = |x-1| < 1 \quad \Longrightarrow \; 0 < x < 2$$

Similarly, the power series is divergent for those x satisfying

$$x < 0 \quad \text{or} \quad x > 2.$$

At the endpoints $x = 0$, $x = 2$ we find

- ($x = 0$) the power series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is convergent by the Alternating Series Test.

- ($x = 2$) the power series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent by the p-series test.

Hence, the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges on the interval $0 \leq x < 2$ and diverges otherwise.

Check your understanding

Determine the largest interval on which the power series converges.

1.
$$\sum_{n=0}^{\infty} \frac{x^n}{3^n}$$

2.
$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{5^{n+1}}$$

Remark 1.1. The largest interval on which a power series centred at c converges is always an *interval centred at* c. This is where the terminology comes from.

Definition 1.2. Let $\sum_{n=0}^{\infty} c_n(x-c)^n$ be a power series. Then, the largest interval on which the power series converges is called the **interval of convergence**.

There are three possibilities for the interval of convergence of a power series:

1. the interval of convergence is a single point $x = c$;
2. the interval of convergence is a finite interval of the form

   $$(c - R, c + R), \quad \text{or} \quad [c - R, c + R], \quad \text{or} \quad (c - R, c + R], \quad \text{or} \quad [c - R, c + R)$$

   for some R (the **radius of convergence**)
3. the interval of convergence is $(-\infty, \infty)$

2 Power series representing functions

Given a power series $\sum_{n=0}^{\infty} c_n(x-c)^n$ with interval of convergence I, we can define a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x - c)^n, \quad \text{for } x \text{ in } I.$$

Any function defined in this way as a power series admits the following properties:

Properties of functions defined by power series:

- $f(x)$ is **differentiable** (and therefore continuous) on I and

$$f'(x) = c_1 + 2c_2(x - c) + 3c_3(x - c)^2 + \ldots$$

i.e. the power series can be differentiated term-by-term.

- $f(x)$ is **integrable** and

$$\int f(x)dx = C + c_0(x - c) + \frac{c_1}{2}(x - 1)^2 + \frac{c_2}{3}(x - c)^3 + \ldots$$

i.e. the power can be integrated term-by-term.

Moreover, the series obtained by differentiation/integration are centred at c and have the same radius of convergence as $f(x)$. The endpoints of the interval of convergence need to be given further investigation.

These results will be very useful in giving power series representations of well-known functions.

Example 2.1. 1. Recall that if $|x| < 1$ then

$$1 + \sum_{n=1}^{\infty} x^n = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad}$$

Hence, we have

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{d}{dx}(\underline{\hspace{6cm}})$$

$$= \underline{\hspace{7cm}}$$

2. Consider the function

$$\frac{1}{1-4x^2}$$

If we let $y = 4x^2$ then we can write

$$\frac{1}{1-4x^2} = \frac{1}{1-y} = 1 + y + y^2 + y^3 + \ldots = \sum_{n=0}^{\infty} y^n$$

This power series representation is well-defined whenever

$$|y| < 1 \quad \Leftrightarrow \quad 4|x|^2 < 1 \Leftrightarrow |x| < \frac{1}{2}.$$

3. Observe that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

Hence,

$$\ln(1+x) = \int \frac{1}{1+x}dx = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \ldots)dx$$

$$= \underline{\hspace{7cm}}$$

As $0 = \ln(1)$ we find $C = 0$.

The radius of convergence of $\frac{1}{1+x}$ is $R = 1$ (because $\frac{1}{1+x}$ converges when $-1 < x < 1$), and the radius of convergence of the power series expansion of $\ln(1+x)$ is also $R = 1$. However, when $x = 1$ the series is convergent (by Alternating Series Test), and we determine the limit

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \ldots$$

4. The following series expansion of $\arctan(x)$ was first given by James Gregory, a 17th century Scottish mathematician (who lived < 10 miles from where I grew up!).

Recall that

$$\arctan(x) = \int \frac{1}{1+x^2}dx$$

Now, since

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

we find

$$\arctan(x) = \int \frac{1}{1+x^2} dx = \int 1 - x^2 + x^4 - x^6 + x^8 - \ldots dx$$

$$= \rule{6cm}{0.4pt}$$

Since $\arctan(0) = 0$ we find $C = 0$.

The radius of convergence of $\frac{1}{1+x^2}$ is $R = 1$ so that same is true of the power series representation of $\arctan(x)$. When $x = 1$, the series converges (by Alternating Series Test) so that

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Check your understanding

Determine power series representations of the following functions.

1. $f(x) = \frac{1}{1-4x^4}$

2. $f(x) = \frac{2}{3-x}$ (*Hint: rearrange so the series is in the form* $\frac{a}{1-y}$, *for some* y)

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES

Before the next Lecture please attempt the following problems. One student in class will be randomly chosen (your name will be pulled from *The Jar*) to present your solution. If you are unable to solve the problem then *don't worry*! We will work through it together and you will receive help at those points you have found difficult. It's important for you to make a good attempt at these problems even if you are unable to solve them.

- Determine the centre c, the radius of convergence R and the interval of convergence I.

1. $\sum_{n=0}^{\infty}$
2. $\sum_{n=0}^{\infty} \frac{(-x)^{2n}}{(2n)!}$
3. $\sum_{n=0}^{\infty} \frac{(4-2x)^n}{2n+1}$

- Find a power series representation of the following functions. Determine the interval of convergence.

1. $f(x) = \frac{x}{9-x^2}$
2. $f(x) = \frac{1+x}{1-x}$
3. $f(x) = \frac{3}{x^2-x-2}$ (*Hint: complete the square!*)