Contact: `gwmelvin@colby.edu`

September 7 Summary

Supplementary References:

- *Calculus*, Hughes-Hallet et al, Sections 1.1, 1.3

Keywords: *function, increasing, decreasing, concave up/down, composition, inverse function*

Functions

• A **function** f is a rule that assigns to each input x exactly one output $y = f(x)$. The collection of all inputs is called the **domain of** f; the collection of all possible outputs is called the **range of** f.

What do we care about?

Representations of functions:

- in words
- table
- **graph** i.e. in (x, y)-plane plot the points $y = f(x)$
- **formula** e.g. $f(x) = x^2 + 3x - 5$

Types of functions:

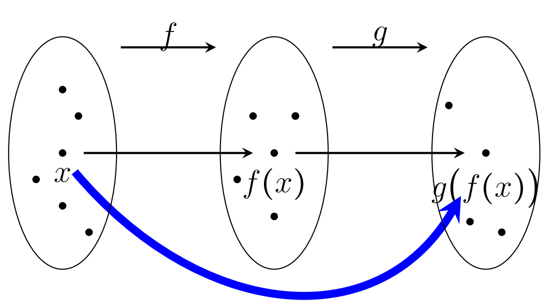
- linear i.e. $f(x) = mx + b$
- exponential (to be discussed 9/10)

Descriptions of functions:

- increasing/decreasing
- concave up/down

• **Remark:** we focus on graph/formula representations; we will develop techniques to plot the graph given a formula.

• Let $f(x), g(x)$ be two functions with the property that the range of $f(x)$ is contained in the domain of $g(x)$ i.e. outputs of $f(x)$ are inputs of $g(x)$. We can form the **composition** $g(f(x))$:



- **Remark:** the notation can be confusing: $g(f(x))$ means

DO f FIRST AND THEN DO g

Example:

1.
$$f: \text{ people in classroom} \to \text{ distance travelled to Colby (miles)}$$
$$g: \text{ distance to Colby (miles)} \to \text{ cost in gas (USD)}$$
The outputs of f are the inputs of g so we can form the composition $g(f(x))$: it is the function that assigns to each person in class the cost (in USD) of gas to get to Colby.

2. Let $f(x) = x^2$, $g(x) = 2x^2 + 3x + 2$, where the domain for both functions is the collection of all real numbers. Then, outputs of f are inputs of g and we can form the composition:
$$g(f(x)) = g(x^2) = 2(x^2)^2 + 3(x^2) + 2 = 2x^4 + 3x^2 + 2$$

- **WARNING:** Be careful! In the examples above:

1. $f(g(x))$ does not make sense: output of g is a number, while the input of f is a person. Can't form the composition in this instance.

2.
$$f(g(x)) = f(2x^2+3x+2) = (2x^2+3x+2)^2 = 4x^4+12x^3+17x^2+12x+4 \neq g(f(x))!!$$

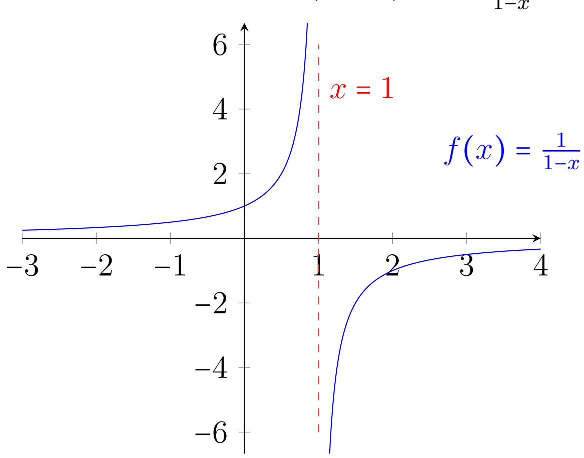
ORDER OF COMPOSITION IS IMPORTANT!

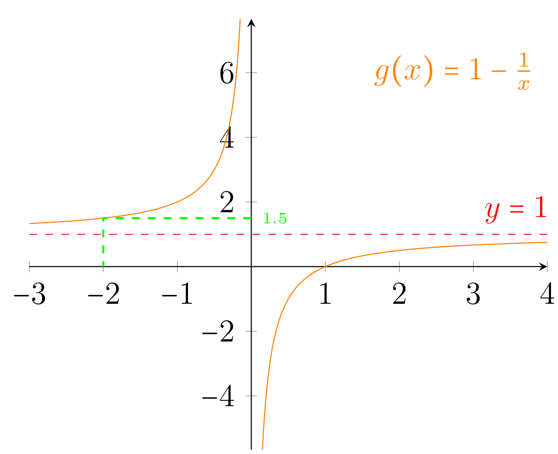
An interesting example:

$$f(x) = \frac{1}{1-x}, \qquad \text{inputs all } x \neq 1$$
$$g(x) = 1 - \frac{1}{x}, \qquad \text{inputs all } x \neq 0$$

Note: outputs of f are inputs of g: outputs of g are inputs of f $\implies$ can form the compositions $f(g(x))$ and $(g(f(x))$.

$$f(g(x)) = f\left(1 - \frac{1}{x}\right) = \frac{1}{1-(1-1/x)} = \frac{1}{\frac{1}{x}} = x \qquad (*)$$
$$g(f(x) = g\left(\frac{1}{1-x}\right) = 1 - \frac{1}{\frac{1}{1-x}} = 1-(1-x) = x \qquad (**)$$





How to interpret $(*)$ and $(**)$?

• $(*)$ Let $y = g(x)$ be an output of g. Then, $(*)$ says that f is the function that outputs x when we input $y = g(x)$ i.e. f is the function that *solves for x for the equation* $y = g(x)$ e.g. if $g(x) = 1.5$ then, to find x satisfying this equation we compute $f(1.5) = -2$. Hence, $g(-2) = 1.5$.

• $(**)$ Let $y = f(x)$ be an output of $f(x)$. Then, $(**)$ says that g is the function that outputs x when we input $y = f(x)$ i.e. g is the function that *solves for x for the equation* $y = f(x)$ e.g. if $f(x) = 3$ then, to find x satisfying this equation we compute $g(3) = 2/3$. Hence, $f(2/3) = 3$.

• Let $f(x)$ be a function. If there is a function $g(x)$ satisfying

- DOMAIN g = RANGE f
- RANGE g = DOMAIN f
- $f(g(x)) = x$ and $(g(f(x)) = x$

then we say that $g(x)$ is the **inverse function of** $f(x)$ and write $f^{-1}(x) = g(x)$.

• **Example:** the function $g(x) = 1 - \frac{1}{x}$ is the inverse function of $f(x) = \frac{1}{1-x}$.
• **Note:**

1. Be careful of the notation introduced above: we are not saying that $f^{-1}(x) = \frac{1}{f(x)}$ e.g. in the example above

$$\frac{1}{f(x)} = \frac{1}{\frac{1}{1-x}} = 1 - x \neq g(x)$$

2. It's not true that every function has an inverse function e.g. the function $f(x) = x^2$, with domain being the collection of all real numbers, does not have an inverse function. We'll see why in class on 9/10.