Math 121C1/E: Single-Variable Calculus
Fall 2018

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Supplementary References:
- *Calculus*, Hughes-Hallet et al, Section 2.3

Keywords: *the derivative function*

The Derivative Function

- **Recall:** Let $f(x)$ be a function. The **derivative function of** $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Some Derivative Rules

- **Power Law:** if $f(x) = x^n$, where $n \geq 0$ is an integer, then $f'(x) = x^{n-1}$
- **Linearity:** Let $f(x), g(x)$ be functions with derivatives $f'(x), g'(x)$.
  - $(f \pm g)'(x) = f'(x) \pm g'(x)$,
  - $(cf)'(x) = cf'(x)$, for any constant c.

Example: Let $f(x) = 3x^3 - x - 1$. Then

$$f'(x) = 3(x^3)' - (x)' - (1)' = 9x^2 - 1$$

This agrees with what we'd compute using the limit definition

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{3(x+h)^3 - (x+h) - 1 - (3x^3 - x - 1)}{h} \\ &= \lim_{h \to 0} \frac{3(x^3 + 3x^2h + 3xh^2 + h^3) - x - h - 1 - 3x^3 + x + 1}{h} \\ &= \lim_{h \to 0} \frac{9x^2h + 9xh^2 + 3h^3 - h}{h} = \lim_{h \to 0} (9x^2 - 1 + 9xh + 3h^2) \\ &= 9x^2 - 1 \end{aligned}$$

Proof of the Power Law: The **Binomial Theorem** states that, for any integer $n \geq 0$,

$$(x+y)^n = 1 \cdot x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-2} x^2 y^{n-2} + \binom{n}{n-1} x y^{n-1} + 1 \cdot y^n$$

where $\binom{n}{k}$ is a **binomial coefficient**, determined using **Pascal's Triangle**:

$$\begin{array}{lccccccccc} n = 0: & & & & & 1 & & & & \\ n = 1: & & & & 1 & & 1 & & & \\ n = 2: & & & 1 & & 2 & & 1 & & \\ n = 3: & & 1 & & 3 & & 3 & & 1 & \\ n = 4: & 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & & \vdots & & & & \end{array}$$

The next row is obtained by adding successive pairs of numbers in the preceding row e.g. the $n = 5$ row is 1 5 10 10 5 1.
For example,

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

the coefficients are obtained by reading off the corresponding row. It's always the case that $\binom{n}{1} = n$.

If $f(x) = x^n$ then

$$f(x+h) = (x+h)^n = x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}xh^{n-1} + h^n$$

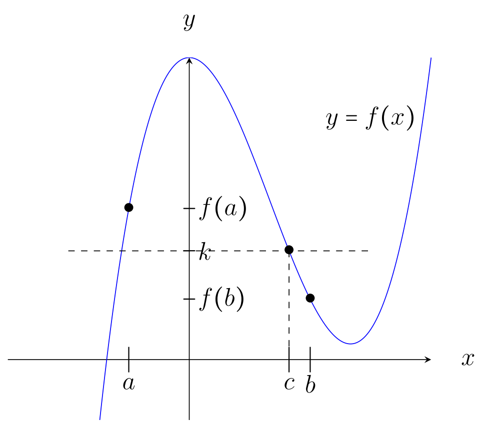
Hence,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \ldots + +\binom{n}{n-1}xh^{n-1} + h^n - x^n}{h}$$

$$= \lim_{h\to 0}\left(\binom{n}{1}x^{n-1} + \ldots + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right) = \binom{n}{1}x^{n-1} = nx^{n-1}$$

Intermediate Value Theorem: (*this should have been introduced when we were discussing continuity*)
• Let $f(x)$ be a continuous function defined on the domain $a \le x \le b$. Suppose that $f(a) \le k \le f(b)$ or $f(a) \ge k \ge f(b)$. Then, there is $a \le c \le b$ such that $f(c) = k$.



Example: Let $f(x) = 5^x - \frac{1}{x}$. Then, $f(1/3) = \sqrt[3]{5} - 3 = -1.29\ldots < 0$ and $f(1) = 5^1 - 1 = 4 > 0$. Let $k = 0$. Hence, by IVT there's some $1/3 \le c \le 1$ so that $f(c) = 0$.