Math 121C1/E: Single-Variable Calculus
Fall 2018

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SUPPLEMENTARY REFERENCES:
- *Calculus*, Hughes-Hallet et al, Section 2.2-2.3

KEYWORDS: *derivative at a point, the derivative function*

The Derivative Function

• **Recall:** Let $f(x)$ be a function, $x = a$ a point in the domain of $f(x)$. Then, the **derivative of** $f(x)$ **at** $x = a$ is

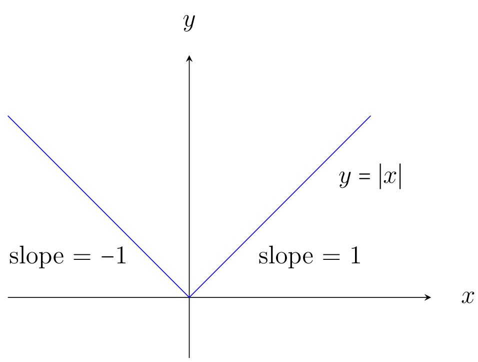
$$f'(a) \overset{def}{=} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

• **Remark:** $f'(a)$ determine the slope of tangent line to graph $y = f(x)$ at $(a, f(a))$.

Example: Let $f(x) = |x|$, $a = 0$. Then

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h) - 0}{h} = \frac{|h|}{h} = \begin{cases} \frac{h}{h} = 1, & h > 0 \\ \frac{-h}{h} = -1, & h < 0 \end{cases}$$

Hence, $\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = 1$ and $\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = -1$. Since these one-sided limits are not equal the derivative of $f(x)$ at $x = 0$ **does not exist**.



• Let $f(x) = 3x^3 - x - 1$. Then, we can compute the following derivatives of $f(x)$ at $x = a$ e.g.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{3h^3 - h - 1 - (-1)}{h} = \lim_{h\to 0}(3h^2 - 1) = -1$$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3(2+h)^3 - (2+h) - 1 - (21)}{h} = \lim_{h\to 0} \frac{35h + 18h^2 + 3h^3}{h}$$
$$= \lim_{h\to 0}(35 + 18h + 3h^2) = 35$$

a	$f'(a)$
2	35
-1	10
0	-1
1	10
2	35

The assignment

$$x \quad \mapsto \quad f'(x)$$

defines the **derivative function** $f'(x)$:

$$f'(x) \stackrel{def}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Example:

1. Let $f(x) = k$, where k is a constant. Then

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim h \to 0 \frac{k-k}{h} = \lim_{h\to 0} 0 = 0$$

   Hence, $f'(x) = 0$.

2. Let $f(x) = 2x + 3$. Then,

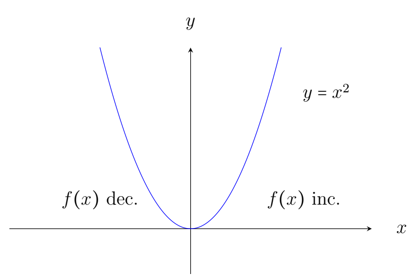
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)+3-2x-3}{h} = \lim_{h\to 0} 2 = 2$$

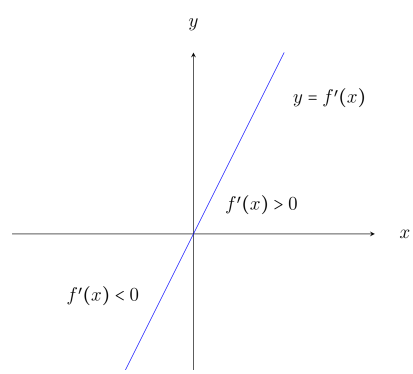
   Hence, $f'(x) = 2$. Generally, if $f(x) = mx + b$ then $f'(x) = m$.

3. Let $f(x) = x^2$. Then,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x$$

   Hence, $f'(x) = 2x$.





- $f'(x) < 0$ implies $f(x)$ decreasing;
- $f'(x) > 0$ implies $f(x)$ increasing;