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September 21 Summary

Supplementary References:
- *Calculus*, Hughes-Hallet et al, Section 2.2

Keywords: *derivative at a point*

Derivative at a point

• **Derivative at a point:** Let $f(x)$ be a function, $x = a$ in the domain of $f(x)$. The quantity

$$\frac{f(a+h)-f(a)}{h}$$

measures the slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$. It determines what we will call the **average rate of change of** $f(x)$ **with respect to** x **on the interval** $[a, a+h]$ (**note:** also allow $h < 0$ so that $a+h < a$). Therefore,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

determines the slope of the tangent line to $y = f(x)$ at $(a, f(a))$.

Example:

1. Let $f(x) = x^3 - 27x + 2$, $a = 3$. First, we note that

$$(3+h)^3 = 27 + 27h + 9h^2 + h^3$$

Then,

$$\frac{f(3+h)-f(3)}{h} = \frac{(3+h)^3 - 27(3+h) + 2 - (27 - 27\cdot 3 + 2)}{h} = \frac{9h^2+h^3}{h} = 9h + h^2$$

This last expression is a continuous function of h so we can evaluate $\lim_{h\to 0} 9h + h^2$ by substituting $h = 0$. Hence,

$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} 9h + h^2 = 0$$

This tells us that the tangent line to $y = x^3 - 27x + 2$ at $x = 3$ is horizontal i.e. has slope 0. This can be interpreted as saying that near to $(3, f(3)) = (3, -52)$ the graph is *locally flat*.

2. Let $g(x) = \frac{1}{x}$, $a = -3$. Then,

$$\frac{g(-3+h)-g(-3)}{h} = \frac{1}{h}\left(\frac{1}{-3+h} - \frac{1}{-3}\right) = \frac{-h}{-3h(-3+h)} = \frac{1}{3(h-3)}$$

Hence, since this last expression is a continuous function of h, and defined at $h = -3$, we can substitute $h = -3$ to get

$$\lim_{h \to 0} \frac{g(-3+h) - g(-3)}{h} = \lim_{h \to 0} \frac{1}{3(h-3)} = \frac{-1}{9}$$

This tells us that the tangent line to $y = \frac{1}{x}$ at $x = -3$ has slope $-1/9$ i.e. near to $(-3, -1/3)$ the graph/function is decreasing.

We can compute the tangent line using **point-slope formula**: given a point (a, b) and slope m, there is a unique line with slope m passing through (a, b) given by equation $y - b = m(x - a)$. Hence, equation of tangent line through $(3, -1/3)$ is

$$y - (-1/3) = \frac{-1}{9}(x+3) \quad \Longrightarrow \quad y = -\frac{x}{9} - \frac{2}{3}$$

- The **derivative of** $f(x)$ **at** $x = a$ is

$$f'(a) \overset{def}{=} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  - $f'(3) = 0$, $g'(-3) = -1/9$.

• Let $f(x)$ be a function, $x = c$ in the domain of $f(x)$. Say that $f(x)$ **is continuous at** $x = c$ if

(A) $\lim_{x \to c} f(x) = L$ exists, and

(B) $L = f(c)$.

If $f(x)$ is continuous for every c in its domain then we say $f(x)$ is **continuous**.

Example:

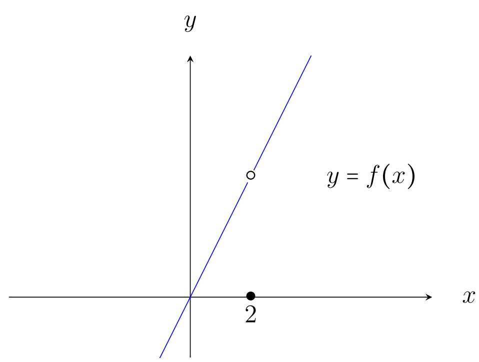
1. $f(x) = 2x$, domain $= \mathbb{R}$. Our computation from 9/14 showed that $\lim_{x \to 2} f(x) = 4$. Since $f(2) = 4$, we have that $f(x)$ is continuous at $x = 2$.

   More generally: using Limit Laws we can show that, for any real number c,

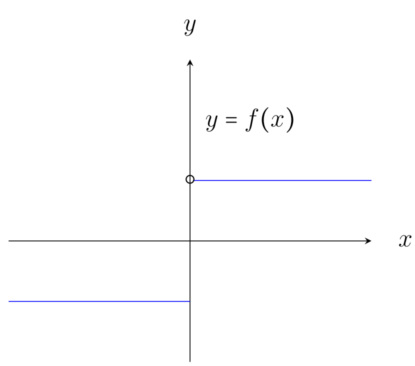
   $$\lim_{x \to c} 2x \overset{LL1}{=} 2 \lim_{x \to c} x \overset{LL6}{=} 2c$$

   Thus, since $f(c) = 2c$, $f(x) = 2x$ is continuous at every point in its domain. Hence, $f(x)$ is continuous.

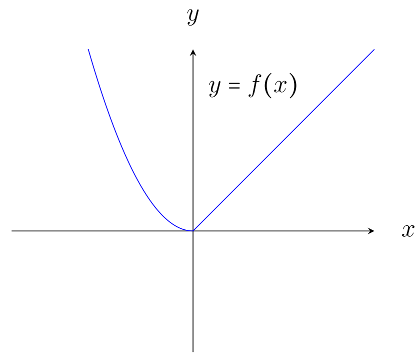
2. Let $f(x) = \begin{cases} 2x, & x \neq 2 \\ 0, & x = 2 \end{cases}$. Then, $\lim_{x \to 2} f(x) = 4$. However, $f(2) = 0 \neq 4$. Hence, $f(x)$ is not continuous at $x = 2$.



3. Consider $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$. In this case, $\lim_{x \to 0^+} f(x) = 1$ while $\lim_{x \to 0^-} f(x) = -1$. Therefore, $\lim_{x \to 0} f(x)$ does not exist and $f(x)$ is not continuous at $x = 0$.



4. Consider $f(x) = \begin{cases} x^2, & x < 0 \\ x, & x \geq 0 \end{cases}$. As x approaches 0, $f(x)$ approaches 0 from both sides. Hence, $\lim_{x \to 0} f(x) = 0$. Since $f(0) = 0$, $f(x)$ is continuous at $x = 0$.



George-given Truths:

• all 'nice' functions - e.g. polynomial, rational, exponential, trigonometric, logarithmic - are continuous everywhere in their domain.

• piecewise functions are continuous, except possibly at jump discontinuities.

Example: Determine $\lim_{x \to 7} \dfrac{x^3 - 7x^2 + 1}{x^2 + 3}$.

Define $f(x) = \frac{x^3 - 7x^2 + 1}{x^2 + 3}$. Then, f is defined for all x. Since $f(x)$ is a rational function, it is continuous everywhere. In particular, $f(x)$ is continuous at $x = 7$ so that

$$\lim_{x \to 7} \frac{x^3 - 7x^2 + 1}{x^2 + 3} = \lim_{x \to 7} f(x) = f(7) = \frac{7^3 - 7 \cdot 7^2 + 1}{7^2 + 3} = \frac{1}{52}$$