Math 121C1/E: Single-Variable Calculus
Fall 2018

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Supplementary References:
- *Calculus*, Hughes-Hallet et al, Section 1.8

Keywords: *limits at infinity, rigorous definition of limit*

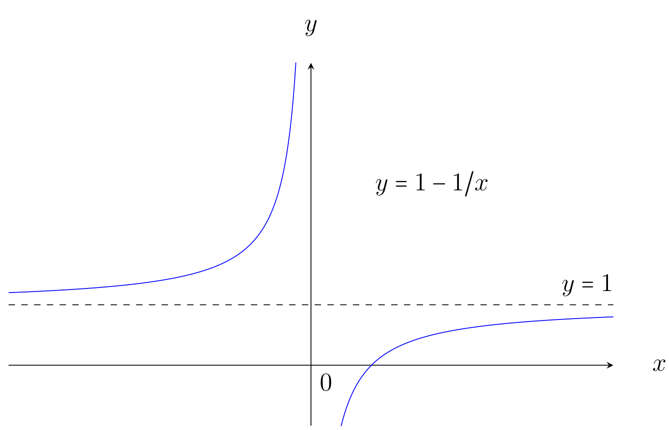
Limits contd.

Example: (Non-existent limits)

Consider $f(x) = \sin(1/x)$, $x \neq 0$.
• if $x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$ then $1/x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$ and $f(x) = 1$ for these values of x.
• if $x = \frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi}, \ldots$ then $1/x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$ and $f(x) = -1$ for these values of x.
• Summary: as we approach $x = 0$ from the right, $f(x)$ oscillates back and forth between 1 and -1 i.e. $f(x)$ does not approach a single value as x gets closer to 0 $\implies \lim x \to 0 f(x)$ DNE.

Other types of limits: limits at infinity
• If $f(x)$ approaches a finite real number L as x gets very large then we write $\lim\limits_{x\to\infty} f(x) = L$.
• If $f(x)$ approaches a finite real number L for $x < 0$ and as $|x|$ gets very large then we write $\lim\limits_{x\to-\infty} f(x) = L$.

Example: Let $f(x) = 1 - \frac{1}{x}$, $x \neq 0$. As x gets very large, $\frac{1}{x}$ becomes very small and $1 - \frac{1}{x}$ gets close to $1 - 0 = 1$. Hence, $\lim\limits_{x\to\infty} f(x) = 1$.
Similarly, for $x < 0$ with $|x|$ very large, the quantity $\frac{1}{x}$ is negative with small absolute value. Hence, $1 - \frac{1}{x}$ gets close to $1 + 0 = 1 \implies \lim\limits_{x\to-\infty} f(x) = 1$ also.



Rigorous definition of limit
• **Recall:** we write $\lim\limits_{x\to c} f(x) = L$ if $f(x)$ *approaches* L as x gets *close*, but not equal, to c. There is some ambiguity in what we mean by *approaches* and *close* in this

definition: *how close do we want $f(x)$ to get to L?* Of course, we want $f(x)$ to get *as close as we like to L*. What this means is the following: I'll write $\lim_{x\to c} f(x) = L$ to signify that, no matter how small a number you give me, I can find values of x close enough to c so that $f(x)$ is within the distance you desire.

Mathematically we write: if you give me any small number $\epsilon > 0$, I can find some distance $\delta > 0$ so that, for those $x \neq c$ within distance δ of c, the values $f(x)$ are within ϵ of L. Now, realising that

$$|a - b| < c \quad \text{means '\textit{the distance between } a \textit{ and } b \textit{ is less than } c'}$$

we can give the rigorous definition of the limit:

- write $\lim_{x\to c} f(x) = L$ if, given any $\epsilon > 0$, there exists $\delta > 0$ so that

$$x \neq c \text{ and } |x - c| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon$$