Math 121C1/E: Single-Variable Calculus
Fall 2018

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September 17 Summary

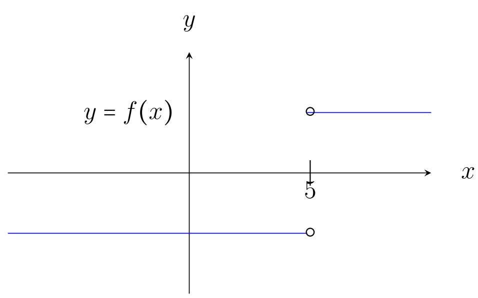
Supplementary References:
- *Calculus*, Hughes-Hallet et al, Section 1.8

Keywords: *one-sided limits, Limit Laws*

Limits contd.

• Let $f(x)$ be a function. Write $\lim_{x\to a} f(x) = L$ if the *values of* $f(x)$ *approach* L *as* x *gets closer and closer,* **but not equal**, *to* a.

Example: Let $f(x) = \frac{|x-5|}{x-5}$, $x \neq 5$



As x gets close to $a = 5$ there are two possibilities for $L \implies \lim_{x\to 5} f(x)$ DNE.

- if x approaches 5 from the right then $f(x)$ approaches $L = 1$; we write

$$\lim_{x\to 5^+} f(x) = 1$$

- if x approaches 5 from the left then $f(x)$ approaches $L = -1$; we write

$$\lim_{x\to 5^-} f(x) = -1$$

Call these **one-sided limits**.

• Even though the limit $\lim_{x\to 5} f(x)$ DNE, the two one-sided limits do exist.

• In general, $\lim_{x\to c} f(x)$ exists if both one-sided limits $\lim_{x\to c^\pm} f(x)$ exist and are equal.

• **Question:** how to determine limits for $f(x)$ if we are unable to graph?

Limit Laws: Let $f(x)$, $g(x)$ be functions. Assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist.

(LL1) $\lim_{x\to c} bf(x) = b\left(\lim_{x\to c} f(x)\right)$, for any constant b.

(LL2) $\lim_{x\to c} f(x) \pm g(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

(LL3) $\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

(LL4) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$.

(LL5) For any constant k, $\lim_{x\to c} k = k$.

(LL6) $\lim_{x\to c} x = c$.

Example:

$$\begin{aligned}
&\lim_{x\to 2} \frac{2x^2+5x+1}{x+4} \\
&= \frac{\lim(2x^2+5x+1)}{\lim(x+4)}, \quad \text{by LL4} \\
&= \frac{2\lim x^2 + 5\lim x + \lim 1}{\lim x + \lim 4}, \quad \text{by LL1, LL2} \\
&= \frac{2(\lim x)^2 + 5\cdot 2 + 1}{2+4}, \quad \text{by LL5, LL6} \\
&= \frac{2\cdot 2^2 + 5\cdot 2 + 1}{2+4}, \quad \text{by LL3} \\
&= \frac{19}{6}
\end{aligned}$$

• **Remark:** the fact that this limit is equal to what we'd obtain by inputting $x = 2$ into the expression and evaluating is a consequence of the fact that $f(x) = \frac{2x^2+5x+1}{x+4}$ is continuous at $x = 2$. More on that tomorrow.

• **Some further non-existent limits:** consider the function $f(x) = \frac{1}{x^2}$. Then, as x gets closer and closer to 0, $f(x)$ grows without bound i.e. there is no finite value L that $f(x)$ approaches as x approaches 0. Hence, $\lim_{x\to 0} f(x)$ DNE.

In this very special case - when $f(x)$ grows without bound as x approaches 0 - we write $\lim_{x\to 0} f(x) = +\infty$. It's important to remember that this **limit does not exist**.